$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Ex:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 5 & 2 & 2 \end{bmatrix}$$

Multiply R_2 and R_3 by a_{11} $\longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$

Multiply R_2 and R_3 by 3 $\longrightarrow \begin{bmatrix} 3 & 2 & 1 \\ 12 & 15 & 6 \\ 15 & 6 & 6 \end{bmatrix}$

Replace
$R_2 \to R_2 + (-a_{21})R_1$
and
$R_3 \to R_3 + (-a_{31})R_1$

$$\longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Replace
$R_2 \to R_2 + (-4)R_1$
and
$R_3 \to R_3 + (-5)R_1$

$$\longrightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & -4 & 1 \end{bmatrix}$$

Multiply R_3 by $a_{11}a_{22} - a_{12}a_{21}$

$$\longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{33} - a_{13}a_{31}) \end{bmatrix}$$

Multiply R_3 by 7

$$\longrightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & -28 & 7 \end{bmatrix}$$

Replace
$R_3 \to R_3 + [-(a_{11}a_{32} - a_{12}a_{31})]R_2$

$$\longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

Replace
$R_3 \to R_3 + (4)R_2$

$$\longrightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & 0 & 15 \end{bmatrix}$$

where

$$\begin{aligned} a_{11}\Delta &= (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad -(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{23} - a_{13}a_{21}) \\ &= a_{11}^2a_{22}a_{33} - a_{11}a_{13}a_{22}a_{31} - a_{11}a_{12}a_{21}a_{33} + a_{12}a_{13}a_{21}a_{31} \\ &\quad -(a_{11}^2a_{23}a_{32} - a_{11}a_{13}a_{21}a_{32} - a_{11}a_{12}a_{23}a_{31} + a_{12}a_{13}a_{21}a_{31}) \\ &= a_{11}(a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} \\ &\quad -a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}) \end{aligned}$$

$$\begin{aligned} \text{So } \Delta &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} \\ &\quad +a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned}$$

Let's try to rewrite this determinant for a 3×3 matrix in a better form:

$$\begin{aligned}
\Delta &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11} \cdot \qquad\qquad - a_{12} \cdot \qquad\qquad + a_{13} \cdot \\
&= a_{11} \cdot \qquad\qquad - a_{12} \cdot \qquad\qquad + a_{13} \cdot
\end{aligned}$$

where A_{ij} is the matrix obtained by ____ .

Ex: Use this formula to compute the determinant of $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 5 & 2 & 2 \end{bmatrix}$. (Same matrix as on p. 1)

Def For $n \geq 2$, the DETERMINANT of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\begin{aligned}
\det A &= \\
&=
\end{aligned}$$

Note: If A is a 4×4 matrix, A_{ij} is a 3×3 matrix. So $\det A$ is defined in terms of the determinant of a 3×3 matrix, which in turn is defined in terms of a 2×2 matrix.