1. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -4 & 1 \\ 2 & 0 & 12 \end{bmatrix}$, if it exists.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ -1 & -4 & 1 & 0 & 1 & 0 \\ 2 & 0 & 12 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & -6 & 12 & -2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 6 & -12 & 2 & 0 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 3 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & -6 & 8 & 6 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 3 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -4/3 & -1 & 1/6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 6 & -1/2 \\ 0 & 1 & 0 & -7/3 & -2 & 1/6 \\ 0 & 0 & 1 & -4/3 & -1 & 1/6 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 8 & 6 & -1/2 \\ -7/3 & -2 & 1/6 \\ -4/3 & -1 & 1/6 \end{bmatrix}$

But why does this method work? [Rhetorical Question... By the end of this worksheet, you will have proved why it works.]

<u>Def</u> An <u>**elementary matrix** E</u> is one that is obtained by performing <u>a single elementary row operation</u> on the Identity matrix.

<u>Ex</u>:

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_3$ $\Rightarrow$ Elementary Matrix $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $3R_1 \leftrightarrow R_1$ $\Rightarrow$ Elementary Matrix $E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $-2R_3 + R_2 \leftrightarrow R_2$ $\Rightarrow$ Elementary Matrix $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Ex</u>: Given a general 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and the three elementary matrices defined above, compute the following products.

(a). $E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(b). $E_2 A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(c). $E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

[Go on to answer the next two questions (any maybe start filling in the blanks of the Facts and Proof).]

Which elementary row operation transforms A into the resulting E_1A, E_2A, E_3A, respectively?

How do these elementary row operations compare with the ones used to transform I into E_1, E_2, E_3, respectively?

<u>Fact</u> The result of performing an elementary row operation on a $m \times n$ matrix A can be written as the product ___EA___ where E is the $m \times m$ elementary matrix corresponding to performing ___the same row operation___ on the identity matrix I_m. [Given w/o proof – but should be clear from previous example.]

<u>Fact</u> Each elementary matrix is invertible. [Given w/o proof – but E^{-1} is clearly found by performing the operation necessary to take E back to the identity I.]

<u>Theorem</u> An $n \times n$ matrix is invertible if and only if A is row equivalent to I_n.

<u>Proof</u> $\Rightarrow$: Let A be an ___invertible___ $n \times n$ matrix. [Show that it is row equivalent to I_n.

i.e. Show that the unique RREF is ___I_n___ .]

Then for each ___$\mathbf{b}$___ in $\mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has a (unique) solution (by Theorem ___5 on p. 104___).

Thus, A has a ___pivot___ in every row, so there are n pivots.

Since A is $n \times n$, there are pivots in ___each column___ , as well.

Thus the n pivot positions are along the diagonal.

Therefore, the ___reduced echelon___ form of A is I_n, i.e. $A \sim I_n$.

$\Leftarrow$: [Stop here. We will prove this direction together in class.]

Note: From the previous proof, we saw that the sequence of row operations that reduce A to I can be written in the product form $(E_p \dots E_2 E_1)A = I$ for the elementary matrices $E_1, E_2, \dots, E_p$ corresponding to the row operations in order $1, 2, \dots, p$.

Corollary If A is an invertible $n \times n$ matrix, then any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.

Proof Since A is invertible, we have by the previous theorem that $A =$ $(E_p \cdots E_2 E_1)^{-1}$

for some set of elementary matrices corresponding to a sequence of elementary row operations.

Taking the inverse of both sides

$$\Rightarrow \quad \begin{aligned} A^{-1} &= \left((E_p \cdots E_2 E_1)^{-1}\right)^{-1} \\ &= E_p \cdots E_2 E_1 \end{aligned}$$

since the inverse of an inverse matrix returns the original matrix.

Since multiplying by the identity I_n does not change the matrix, multiply the RHS by I_n to obtain

$$A^{-1} = (E_p \cdots E_2 E_1) I_n.$$

In other words, the same sequence of elementary row operations applied in the order $E_1, E_2, \dots E_p$, which reduce A to I_n will reduce I_n to A^{-1}. ■

[Note: This corollary proves why $[A|I] \;\to\; [I|A^{-1}]$.