1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. T is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

<u>Proof</u> Let T be defined as above.

$\Longrightarrow$: Let T be one-to-one. [Show that $T(\mathbf{x}) = \mathbf{0}$ has ____________________.]

By definition of one-to-one, if $\mathbf{b}$ is in $\mathbb{R}^m$, then there is at most one solution $\mathbf{x}$ in $\mathbb{R}^n$ such that _______________

Specifically, since the zero vector $\mathbf{0}$ is in __________ and T is one-to-one,

there is _________________ solution $\mathbf{x}$ in $\mathbb{R}^n$ such that _______________ . [By the definition of one-to-one]

Since T is linear, ____________ is always a solution to $T(\mathbf{x}) = \mathbf{0}$. i.e. $T(\mathbf{0}) = \mathbf{0}$.

Since there is __________________ solution, $\mathbf{x} = \mathbf{0}$ is the *only* solution.

$\Longleftarrow$: Let $T(\mathbf{x}) = \mathbf{0}$ have only the trivial solution. [Show that T is one-to-one.]

BWOC, suppose ________________________ .

Then there exists a vector $\mathbf{b}$ in $\mathbb{R}^m$ and two _____________ vectors $\mathbf{u}$ and $\mathbf{v}$ such that $T(\mathbf{u}) = \mathbf{b}$ and $T(\mathbf{v}) = \mathbf{b}$.

Then

$$
\begin{aligned}
T(\mathbf{u} - \mathbf{v}) &= \underline{\qquad\qquad\qquad} \quad \text{since } T \text{ is linear.} \\
&= \mathbf{b} - \mathbf{b} \\
&= \mathbf{0}
\end{aligned}
$$

i.e. $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, which has only the ____________ solution [by the given statement (see $\Longleftarrow$:)].

So $\mathbf{u} - \mathbf{v} = \mathbf{0}$ is this trivial solution.

$\Rightarrow$ ____________ $\rightarrow\!\!\!\leftarrow$

Therefore, T must be one-to-one. ∎

2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined as $T(\mathbf{x}) = A\mathbf{x}$. Then

(a). T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$.

(b). T is one-to-one iff the columns of A are linearly independent.

<u>Proof</u> Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined as $T(\mathbf{x}) = A\mathbf{x}$

(a). By a previous theorem (sec 1.4),

the columns of A span $\mathbb{R}^m$ iff for __________________ the equation $A\mathbf{x} = \mathbf{b}$ has a ______________ (i.e. at least one solution).

But since $A\mathbf{x} = \mathbf{b}$ is equivalent to the equation _______________ , the statement becomes:

The columns of A span $\mathbb{R}^m$ iff for each $\mathbf{b}$ in $\mathbb{R}^m$ the equation _______________ has at least one solution.

Therefore, by definition of ___________ , T is onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$. ■

(b). From sec. 1.7, the columns of A are linearly independent iff $A\mathbf{x} = \mathbf{0}$ has ____________________________ .

$\Rightarrow$ The columns of A are linearly independent iff _______________ has only the trivial solution.

From the previous theorem, T is ________________ iff $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Therefore, combining the last 2 statements:

T is one-to-one iff the columns of A are _________________________ . ■

[Now we can complete the Invertible Matrix Theorem.]