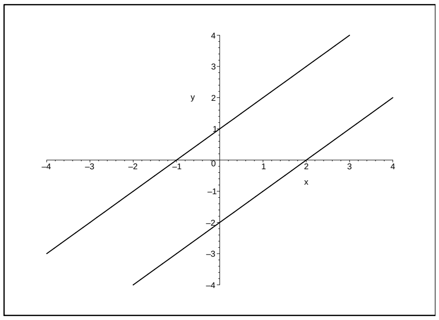
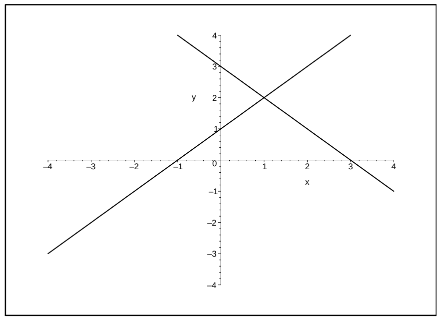
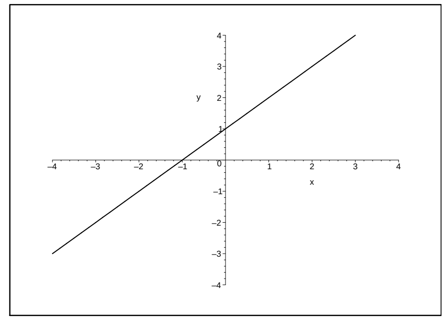
Graph 2 lines in $\mathbb{R}^2$ and only 3 things can happen:







lines are parallel | ***lines intersect at exactly one point*** | ***lines are the same***

Given **equations** for 2 lines,

these graphical possibilities show the 3 possibilities for a solution to both equations

$$\begin{array}{rcrcr} y &=& x &+& 1 \\ y &=& x &-& 2 \end{array} \qquad \begin{array}{rcrcr} y &=& x &+& 1 \\ y &=& -x &+& 3 \end{array} \qquad \begin{array}{rcrcr} x &-& y &=& -1 \\ -2x &+& 2y &=& 2 \end{array}$$

no solution

no point
makes both eqns true

unique solution

(1,2) is only point that
makes both eqns true

infinitely many solutions

all points on the line
make both eqns true

inconsistent | ***consistent*** | ***consistent and dependent***

<u>Def</u> A ____________________ in n variables $x_1, x_2, x_3, ..., x_n$ is of the form

where $a_1, a_2, a_3, ..., a_n$ and b are ____________________________ and $a_1, a_2, a_3, ..., a_n$ are the _______________ .

Ex: $y = x + 1$ Ex: $2\sqrt{3}x_1 - x_2 + (10 - 3i)x_4 = 7$ Ex: $x_1 = 3(x_2 + 4x_3)$

Def A **system of linear equations**

Ex: (3 examples on page 1) Ex: $\begin{array}{rcrcrcr} 3x_1 & & & - & x_3 & = & 10 \\ x_1 & + & x_2 & + & 4x_3 & = & 6 \end{array}$ Ex: $\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ 2x_1 & - & x_2 & + & x_3 & = & 13 \\ & & -4x_2 & + & 5x_3 & = & 33 \end{array}$

Def A ________________ is a list of numbers $(s_1, s_2, s_3, ..., s_n)$ that make all of the equations of a system ________ when substituted for $x_1, x_2, x_3, ..., x_n$, respectively.

Def The ____________ is the set of all possible solutions to a system.

Def Two linear systems are ____________ if they have the same solution set.

Ex: $(3, 7, -1)$ is a solution to the system $\begin{array}{rcrcrcr} 3x_1 & & & - & x_3 & = & 10 \\ x_1 & + & x_2 & + & 4x_3 & = & 6 \end{array}$

Ex: $\{(k, k+1) | k \in \mathbb{R}\}$ is the solution set to the system $\begin{array}{rcrcr} x & - & y & = & -1 \\ -2x & + & 2y & = & 2 \end{array}$

Def A system of linear equations is called

- ____________ if it has no solutions.
- ____________ if it has either one or infinitely many solutions.

  Furthermore, if it has infinitely many solutions the system is called ____________ .

Important Note:

<u>Def</u> A **<u>matrix</u>** is a rectangular array of elements (often numbers)

An $m \times n$ matrix has m ___________ and n ______________

A system of linear equations can be represented by two types of matrices:

<u>Ex</u>: <u>Linear System</u>

1. <u>Coefficient Matrix</u>

coefficients of each variable form the columns

2. <u>Augmented Matrix</u>

the coefficient matrix with an additional column containing the RHS constants

$$\begin{array}{rcrcrcr} 3x_1 & & & + & 3x_3 & = & -6 \\ x_1 & + & 2x_2 & - & x_3 & = & 0 \\ 2x_1 & - & x_2 & + & 5x_3 & = & 1 \end{array}$$

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3 Operations on systems/matrices that yield an equivalent system/matrix $\Longrightarrow$

(a). **<u>Interchange</u>**: Interchange 2 equations/rows

(b). **<u>Scaling</u>**: Multiply the entire equation/row by a nonzero constant

(c). **<u>Replacement</u>**: Replace one equation/row by the sum of itself and a multiple of another equation/row

Ex:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ & & -4x_2 & + & 5x_3 & = & 33 \\ 2x_1 & - & x_2 & + & x_3 & = & 13 \end{array} \qquad \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

Ex:

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & -4 & 5 & 33 \\ 2 & -1 & 1 & 13 \end{bmatrix}$$

1. Interchange equations ② and ③

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ 2x_1 & - & x_2 & + & x_3 & = & 13 \\ & & -4x_2 & + & 5x_3 & = & 33 \end{array} \qquad \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

1. Interchange rows 2 and 3

2. Replace equation ② with: $-2 \cdot$ ①$+$ ②

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ & & -5x_2 & - & 5x_3 & = & -15 \\ & & -4x_2 & + & 5x_3 & = & 33 \end{array} \qquad \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

2. Replace row 2 with: $-2\cdot$row1$+$row2

3. Scale equation ② by multiplying by $-\frac{1}{5}$

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & + & 3x_3 & = & 14 & \quad ① \\ & & x_2 & + & x_3 & = & 3 & \quad ② \\ & & -4x_2 & + & 5x_3 & = & 33 & \quad ③ \end{array}$$

3. Scale row 2 by multiplying by $-\frac{1}{5}$

4. Replace equation ③ with: $4 \cdot$ ② + ③

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & + & 3x_3 & = & 14 & \quad ① \\ & & x_2 & + & x_3 & = & 3 & \quad ② \\ & & & & 9x_3 & = & 45 & \quad ③ \end{array}$$

4. Replace row 3 with: $4 \cdot \text{row2} + \text{row3}$

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5. Scale equation ③ by multiplying by $\frac{1}{9}$

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & + & 3x_3 & = & 14 & \quad ① \\ & & x_2 & + & x_3 & = & 3 & \quad ② \\ & & & & x_3 & = & 5 & \quad ③ \end{array}$$

5. Scale row 3 by multiplying by $\frac{1}{9}$

6. Replace equation ② with: $-1 \cdot$ ③ + ②

Replace equation ① with: $-3 \cdot$ ③ + ①

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & & & = & -1 & \quad ① \\ & & x_2 & & & = & -2 & \quad ② \\ & & & & x_3 & = & 5 & \quad ③ \end{array}$$

6. Replace row 2 with: $-1 \cdot \text{row3} + \text{row2}$

Replace row 1 with: $-3 \cdot \text{row3} + \text{row1}$

7. Replace equation ① with: $-2 \cdot$ ② + ①

$$\begin{array}{rrrcrl} x_1 & & & = & 3 & \quad ① \\ & x_2 & & = & -2 & \quad ② \\ & & x_3 & = & 5 & \quad ③ \end{array}$$

7. Replace row 1 with: $-2 \cdot \text{row2} + \text{row1}$