Name: Key

Exam 2

Math 362 Linear Algebra – Crawford

15 November 2017

Books and notes (in any form) are not are allowed. You may use a calculator – but please indicate when you use the matrix functions on the calculator. Put all of your work and answers on the separate paper provided and staple this cover sheet on top. Show all your work for credit. ***Good luck!***

Calculator # ____________

1. (12 pts) Determine whether the following transformation is a linear transformation. If it is a linear transformation, then find the standard matrix that implements the mapping. If it is not a linear transformation, then clearly show which properties of linear transformations are violated.

$T(x_1, x_2, x_3) = (2x_1 - 3x_3, 4x_3, x_2x_3)$

2. (12 pts) Let H be the set of points on the line $x + 2y = 0$. Determine whether H is a subspace of $\mathbb{R}^2$. If it is not a subspace, clearly show **all** *subspace* properties that do not hold. [You may find it helpful to let $y = t$ and write the points on the line as an ordered pair (or vector) with the parameter t.]

3. (12 pts) Given the subspace $H = \left\{ \begin{bmatrix} a + 5b - 4c - 3d + e \\ b - 2c + d \\ e \end{bmatrix} : a, b, c, d, e \text{ in } \mathbb{R} \right\}$.

(a). Find a basis for H.

(b). State the dimension of H.

4. (12 pts) Given the following matrix, find the characteristic equation and eigenvalues.

$$\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

5. (20 pts) Given that the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$ has eigenvalues $\lambda = 3$ and -3, diagonalize the matrix A, if possible. Clearly state the diagonal matrix D and the matrix P [You do ***not*** need to find P^{-1}.]. If it is not possible to diagonalize A, clearly explain why.

6. (12 pts) Determine whether the following statements are true or false. If the statement is false, correct the statement and/or clearly explain why it is false. [You may answer these questions on this cover sheet, if you would like.]

(a). A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

False. A plane through the origin is a two-dimensional subspace of $\mathbb{R}^3$

(b). If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in a vector space V and $\dim V = p$, then $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a basis for V.

False S must also be linearly independent.

(c). If A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then the solution is unique for each $\mathbf{b}$.

True

(d). For an $n \times n$ matrix H, if $H\mathbf{x} = \mathbf{c}$ is inconsistent for some $\mathbf{c}$ in $\mathbb{R}^n$, then $H\mathbf{x} = \mathbf{0}$ will have no solution.

False $H\vec{x} = \vec{0}$ will have a nontrivial solution.
(in fact, infinitely many)

7. (22 pts) Prove 2 of the following. Clearly state theorems and properties that you use.
BONUS: You may do (or attempt) all four options and each will be graded out of 11 points. Whichever two you score higher on will be your base grade. Any points from the third problem will be cut in third and added to your base grade.

(a). (Old) Let λ be an eigenvalue of an invertible matrix A. Prove that $\lambda^{-1} = \frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(b). (New)Suppose the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose the $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, 2, \ldots, p$. Show that T is the zero transformation. That is, show that $T(\mathbf{x}) = \mathbf{0}$ for any vector $\mathbf{x}$ in $\mathbb{R}^n$.

(c). (New)Prove both of the following:
(i) If A is invertible and $AB = BA$, then $BA^{-1} = A^{-1}B$.
(ii) If A is invertible, then A^TA is also invertible and $A^{-1} = (A^TA)^{-1}A^T$.

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1. $T(x_1, x_2, x_3) = (2x_1 - 3x_3,\ 4x_3,\ x_2 x_3)$

$T(0,0,0) = (0, 0, 0)$

Let $\vec{u} = (x_1, x_2, x_3)$ & $\vec{v} = (y_1, y_2, y_3)$

$\Rightarrow T(\vec{u}+\vec{v}) = T(x_1+y_1,\ x_2+y_2,\ x_3+y_3)$

$= (2(x_1+y_1) - 3(x_3+y_3),\ 4(x_3+y_3),\ (x_2+y_2)(x_3+y_3))$

$= (2x_1 + 2y_2 - 3x_3 - 3y_3,\ 4x_3 + 4y_3,\ x_2x_3 + x_2y_3 + y_2x_3 + y_2y_3)$

$\neq \underbrace{T(\vec{u}) + T(\vec{v})}$

$= (2x_1 - 3x_3,\ 4x_3,\ x_2x_3) + (2y_1 - 3y_3,\ 4y_3,\ y_2y_3)$ Not the same

$= (2x_1 + 2y_1 - 3x_3 - 3y_3,\ 4x_3 + 4y_3,\ x_2x_3 + y_2y_3)$

T(cu)=T(cx1,cx2,cx3)=(2cx1-3cx3,4cx3,c^2x2x3) ≠ cT(u) = (2cx1-3cx3,4cx3,cx2x3)

2. H: Set of all pts on the line $x + 2y = 0 \Rightarrow x = -2y$

ie $H = \left\{ \begin{bmatrix} -2t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}$ Let $y = t \Rightarrow x = -2t$

$\Rightarrow t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

ie $H = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

Then since $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ & $H = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$,

H is a subspace of $\mathbb{R}^2$

OR

0. $H \subset \mathbb{R}^2$ since the vectors are in $\mathbb{R}^2$

1. Let $t = 0 \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in H$ ✓

2. Let $\vec{u} \in H$ & $\vec{v} \in H$ ie $\vec{u} = \begin{bmatrix} -2t \\ t \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2s \\ s \end{bmatrix}$

$\Rightarrow \vec{u} + \vec{v} = \begin{bmatrix} -2t - 2s \\ t + s \end{bmatrix} = \begin{bmatrix} -2(t+s) \\ (t+s) \end{bmatrix} \in H$ ✓

3. Let $\vec{u} \in H$ ie $\vec{u} = \begin{bmatrix} -2t \\ t \end{bmatrix}$ & c be a scalar

$\Rightarrow c\vec{u} = c\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2ct \\ ct \end{bmatrix} = \begin{bmatrix} -2(ct) \\ ct \end{bmatrix} \in H$ ✓

$\therefore$ H is a subspace of $\mathbb{R}^2$

3. $H = \left\{ \begin{bmatrix} a + 5b - 4c - 3d + e \\ b - 2c + d \\ e \end{bmatrix} : a, b, c, d, e \in \mathbb{R} \right\}$

(a) $a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix} + d\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + e\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & +6 & -8 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$H = \text{Col } A$ So a basis is $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$

pivot colums of orig A

(b) $\boxed{\dim H = 3}$

4. $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \cdot \det(\ ) - 0 \det(\ ) + (2-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix}$

$= (2-\lambda)[(-1-\lambda)(4-\lambda) - (-3)(0)]$

$= \boxed{(2-\lambda)(-1-\lambda)(4-\lambda) = 0}$

$\boxed{\lambda = 2, -1, 4}$

5. $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

$\underline{\lambda=3}$ $A - 3I = \begin{bmatrix} -2 & 2 & -2 \\ -2 & 2 & -2 \\ -6 & 6 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$ $\Rightarrow$ $x_1 = x_2 - x_3$

x_2, x_3 free $\quad x_2 = x_2 \quad \Rightarrow \vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$x_3 = x_3$

$\mathcal{B}(\lambda=3) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda=-3}$ $A + 3I = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & -2 \\ -6 & 6 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - \frac{1}{3}x_3 = 0$ $\qquad x_1 = \frac{1}{3}x_3$

$x_2 - \frac{1}{3}x_3 = 0$ $\Rightarrow$ $x_2 = \frac{1}{3}x_3$ $\Rightarrow \vec{x} = x_3 \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$

x_3 free $\qquad x_3 = x_3$

$\mathcal{B}(\lambda=-3) = \left\{ \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix} \right\}$ or any scalar multiple of this vector $\quad$ use $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$$\boxed{D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}}$$

7 (a) Proof
Let λ be an eigenvalue of an invertible matrix A.

[Show that λ^{-1} is an eigenvalue of A^{-1}]

$\Rightarrow \quad A\vec{x} = \lambda\vec{x}$ for nonzero vector $\vec{x}$. [Show $A^{-1}\vec{x} = \lambda^{-1}\vec{x}$ for nonzero $\vec{x}$]

$\Rightarrow \quad A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$

$I\vec{x} = \lambda A^{-1}\vec{x}$

$\vec{x} = \lambda A^{-1}\vec{x}$ But since A is invertible, then $\lambda \neq 0$

So divide by λ

$\Rightarrow \quad \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$

ie $A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$ for the same nonzero vector $\vec{x}$ & scalar $\frac{1}{\lambda}$

$\therefore \quad \frac{1}{\lambda}$ is an eigenvalue of A^{-1} ■

(b) Proof Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ span $\mathbb{R}^n$ and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

Let $T(\vec{v}_i) = \vec{0}$ for $i = 1, 2, \ldots, p$

Let $\vec{x}$ be a vector in $\mathbb{R}^n$

Then $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p$ since $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$ span $\mathbb{R}^n$

$\Rightarrow T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p)$

$= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots + c_pT(\vec{v}_p)$ since T is a linear trans.

$= c_1\vec{0} + c_2\vec{0} + \ldots + c_p\vec{0}$

$= \vec{0}$

ie $T(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$ ■

7(c) (i) Proof Let A be invertible.

Let $AB = BA$ [Show $BA^{-1} = A^{-1}B$]

$\Rightarrow ABA^{-1} = BAA^{-1}$

$ABA^{-1} = B$

$\Rightarrow A^{-1}ABA^{-1} = A^{-1}B$

$\Rightarrow BA^{-1} = A^{-1}B$ ∎

(ii) Proof Let A be invertible

Then A^T is invertible by IMT

$\Rightarrow A^TA$ is invertible since the product of invertible matrices are also invertible.

$(A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T$ by properties of inverses

$= A^{-1}I$ by " " "

$= A^{-1}$ ∎