Exam 1 Key

1. $A = \begin{bmatrix} 1 & 0 & -2 \\ 4 & -3 & 2 \end{bmatrix} \qquad \begin{bmatrix} -2 & 3 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$

(a) $AB = \begin{bmatrix} -4 & 5 \\ -6 & 7 \end{bmatrix}$ $\quad \underbrace{2\times 3 \quad 3\times 2}_{2\times 2}$

$\det(AB) = (-4)(7) - 5(-6)$
$= -28 + 30 = \boxed{2}$

(b) $B^T - A = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ 4 & -3 & 2 \end{bmatrix}$

$= \boxed{\begin{bmatrix} -3 & 0 & 3 \\ -1 & 4 & -3 \end{bmatrix}}$

2 (a) $\left[\begin{array}{ccc|c} 3 & 5 & -4 & b_1 \\ -3 & -2 & 4 & b_2 \\ 6 & 1 & -8 & b_3 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 3 & 5 & -4 & b_1 \\ 0 & 3 & 0 & b_1+b_2 \\ 0 & -9 & 0 & -2b_1+b_3 \end{array}\right]$

$\xrightarrow{3R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 3 & 5 & -4 & b_1 \\ 0 & 3 & 0 & b_1+b_2 \\ 0 & 0 & 0 & 3(b_1+b_2)-2b_1+b_3 \end{array}\right]$

$\vec{b}$ is in the span of the col^{ns} if the system is consistent

Consistent iff $3b_1 + 3b_2 - 2b_1 + b_3 = 0$

$\boxed{b_1 + 3b_2 + b_3 = 0}$

The span contains all vectors in $\mathbb{R}^3$ that are in the plane
$x + 3y + z = 0$

(b) No, the columns do not span $\mathbb{R}^3$ since the system $A\vec{x} = \vec{b}$ is not consistent for all $\vec{b}$.

(c) No, the vectors are in $\mathbb{R}^3$, so they can't span $\mathbb{R}^2$.

3. $\begin{bmatrix} 1 & 3 & -1 \\ -1 & -6 & 4 \\ 4 & 7 & -1 \end{bmatrix} \xrightarrow[-4R_1+R_3\to R_3]{R_1+R_2\to R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 3 \\ 0 & -5 & 3 \end{bmatrix}$

$\xrightarrow{-\frac{1}{3}R_2\to R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & 3 \end{bmatrix} \xrightarrow[5R_2+R_3\to R_3]{} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$

OR

$\to$ RREF (Calculator) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Pivot in every row.
- $A\vec{x}=\vec{0}$ has only the trivial sol^n

$\Rightarrow$ $\boxed{\text{Linearly Indep.}}$

4. (a)

$\begin{array}{c|ccc} & E & W & C \\ \hline E & .44 & .18 & .25 \\ W & .24 & .34 & .10 \\ C & .32 & .48 & .65 \end{array}$

(b)

$.44x_1 + .18x_2 + .25x_3 = x_1$

$.24x_1 + .34x_2 + .10x_3 = x_2$

$.32x_1 + .48x_2 + .65x_3 = x_3$

$\downarrow$

$-.56x_1 + .18x_2 + .25x_3 = 0$

$.24x_1 - .66x_2 + .10x_3 = 0$

$.32x_1 + .48x_2 - .35x_3 = 0$

$\boxed{\left[\begin{array}{ccc|c} -.56 & .18 & .25 & 0 \\ .24 & -.66 & .10 & 0 \\ .32 & .48 & -.35 & 0 \end{array}\right]} \leftarrow$

5. $A = \begin{bmatrix} 2 & x \\ 3 & -2 \end{bmatrix}$ (a) $\det(A) = (2)(-2) - x(3)$

$= -4 - 3x = 0$

Not invertible if $\det(A) = 0$

$3x = -4$

$\boxed{x = -\frac{4}{3}}$

(b) $A = A^{-1} \Rightarrow \frac{1}{-4-3x} \begin{bmatrix} -2 & -x \\ -3 & 2 \end{bmatrix}$

By observation $\Rightarrow \frac{1}{-4-3x} = -1$ for $A = A^{-1}$

$4 + 3x = 1 \Rightarrow 3x = -3$

$\boxed{x = -1}$

OR $AA^{-1} = I \Rightarrow AA = I$

$\begin{bmatrix} 2 & x \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & x \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4+3x & 0 \\ 0 & 3x+4 \end{bmatrix}$

$4 + 3x = 1 \Rightarrow 3x = -3 \Rightarrow \underline{x = -1}$

6. $\begin{vmatrix} 3 & 2 & 0 & 1 \\ -1 & 2 & 5 & 0 \\ 2 & -2 & 0 & 3 \\ 0 & 1 & 0 & -4 \end{vmatrix} = -5 \begin{vmatrix} 3 & 2 & 1 \\ 2 & -2 & 3 \\ 0 & 1 & -4 \end{vmatrix}$

$= -5 \left(3 \begin{vmatrix} -2 & 3 \\ 1 & -4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + 0 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} \right)$

$= -5 \left[3(8-3) - 2(-8-1) \right]$

$= -5 \left[3(5) - 2(-9) \right]$

$= -5 \left[15 + 18 \right]$

$= -5(33)$

$= \boxed{-165}$

7 (a) Proof: Let $\{\vec{v}_1, \vec{v}_2\}$ be a linearly <u>independent</u> set in $\mathbb{R}^n$.

Then [Show that $\{\vec{v}_1, \vec{v}_1+\vec{v}_2\}$ is also linearly independent.]

(*) $\left\{ \begin{array}{l} c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \\ \text{only for } c_1 = c_2 = 0 \end{array} \right.$ [Show $d_1\vec{v}_1 + d_2(\vec{v}_1+\vec{v}_2) = \vec{0}$ only if $d_1 = d_2 = 0$]

Consider $d_1\vec{v}_1 + d_2(\vec{v}_1+\vec{v}_2) = \vec{0}$ [Show $d_1 = d_2 = 0$]

$\Rightarrow (d_1+d_2)\vec{v}_1 + d_2\vec{v}_2 = \vec{0}$

But since $\vec{v}_1$ & $\vec{v}_2$ are linearly independent,

then $d_1 + d_2 = 0$ and $d_2 = 0$ (See (*))

back subs

$\Rightarrow d_1 = 0$

<u>ie</u> $d_1\vec{v}_1 + d_2(\vec{v}_1+\vec{v}_2) = \vec{0}$ only if $d_1 = d_2 = 0$

$\therefore \{\vec{v}_1, \vec{v}_1+\vec{v}_2\}$ is linearly independent. ■

(b) Proof: Let A be invertible. [Show A^T is invertible & $(A^T)^{-1} = (A^{-1})^T$]

Then A^{-1} exists s.t. $AA^{-1} = I$ and $A^{-1}A = I$

Take the transpose $\Rightarrow (AA^{-1})^T = I^T$ and $(A^{-1}A)^T = I^T$

$\Rightarrow \underbrace{(A^{-1})^T}_{D} A^T = I$ $\quad A^T\underbrace{(A^{-1})^T}_{D} = I$

<u>ie</u> There exists a matrix $D = (A^{-1})^T$ s.t. $DA^T = I$ & $A^TD = I$

$\therefore$ By definition A^T is invertible and $(A^T)^{-1} = D = (A^{-1})^T$ ■

(c) Proof. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix.
Suppose B has no col^{n} of all zeros.
Let the 2^{nd} column of AB be all zeros. (*)

[Show $A\vec{x} = \vec{0}$ has a nontrivial sol^{n}]

$$AB = A[\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_p]$$
$$= [A\vec{b}_1 \ \underset{\uparrow}{A\vec{b}_2} \cdots A\vec{b}_p]$$

$\Rightarrow \quad A\vec{b}_2 = \vec{0}$ from (*)

But since B has no col^{n} of all zeros, then $\vec{b}_2 \neq \vec{0}$.

$\therefore$ The homog eqn $A\vec{x} = \vec{0}$ has a nontrivial sol^{n}, namely $\vec{x} = \vec{b}_2$ ∎