

Interchange, Scaling, and Replacement are called elementary row operations for matrices.

DEF Two matrices are row equivalent if there is a sequence of elementary row operations that **transforms one matrix into the other**.

DEF A leading entry of a row is the left-most nonzero entry in that row.

EX: (from previous worksheet)
$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & -4 & 5 & 33 \\ 2 & -1 & 1 & 13 \end{bmatrix}$$

We found a row equivalent matrix of the one above that was in a “good” form (step 4 of previous worksheet):

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 9 & 45 \end{bmatrix}$$

But we went further to find a row equivalent matrix in an even “better” form (step 7):

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

EX:
$$\begin{bmatrix} -8 & -4 & -6 & -2 & 4 \\ 0 & 0 & 3 & 6 & 3 \\ 4 & 2 & 1 & 0 & -4 \\ 0 & 0 & 2 & 1 & 2 \end{bmatrix} \implies \begin{bmatrix} 4 & 2 & 3 & 1 & -2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & -\frac{5}{4} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

DEF A rectangular matrix is in (row) echelon form (REF) if it has the following 3 properties: [Guess]

1. Any row of all zeros are below the nonzero rows.
2. The position of the leading entry in each row is to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

DEF Furthermore, it is in (row) reduced echelon form (RREF) if these 2 additional properties hold:

4. The leading entry in each nonzero row is 1.
5. Each leading entry (1) is the only nonzero entry in its column. (i.e. 0's above and below the leading entry)

THEOREM The Reduced (Row) Echelon Form is unique for any given matrix.

DEF A pivot position in a matrix is the location of a leading 1 in reduced echelon form.

DEF A pivot column is a column that contains a pivot position.

DEF A pivot is a nonzero number in the pivot position used to create zeros in the rows above and below.

Row Reduction Algorithm
(variant of Gaussian Elimination)

FORWARD PHASE

(to echelon form)

Step 1

Locate the leftmost nonzero column and note:

- This is a pivot column
- The pivot position is at the top of this column

Ex:

$$\begin{array}{rccccrcr} & -5x_2 & + & x_3 & + & x_4 & = & 5 \\ 2x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & = & 11 \\ x_1 & - & 2x_2 & + & 2x_3 & + & 2x_4 & = & 8 \\ 5x_1 & & & & & + & 2x_4 & = & 20 \end{array}$$

$$\begin{bmatrix} 0 & -5 & 1 & 1 & 5 \\ 2 & 1 & 3 & 3 & 11 \\ 1 & -2 & 2 & 2 & 8 \\ 5 & 0 & 0 & 2 & 20 \end{bmatrix}$$

Step 2

Choose a nonzero number in this column to be the pivot.

- Choose wisely
- If necessary, interchange rows to move it to the pivot position
- (optional) Scale row to get a 1 in the pivot position.

Step 3

Use Row Operations to get all zero entries below the pivot

Step 4

Ignore/Cover all rows above and including the pivot position.

Repeat steps 1-4 on the submatrix until echelon form attained.

BACKWARD PHASE
(to *reduced* echelon form)

Step 5

Locate the rightmost pivot.

- (a). Scale row to make pivot = 1.
- (b). Use Row Operations to get zero entries above the pivot.
- (c). Locate the next rightmost pivot. Repeat steps 5(a)-5(b) until reduced echelon form is attained.

$$\begin{bmatrix} 1 & -2 & 2 & 2 & 8 \\ 0 & 5 & -1 & -1 & -5 \\ 0 & 0 & -8 & -6 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l}
 \text{Linear System} \\
 \begin{array}{r}
 -5x_2 + x_3 + x_4 = 5 \\
 2x_1 + x_2 + 3x_3 + 3x_4 = 11 \\
 x_1 - 2x_2 + 2x_3 + 2x_4 = 8 \\
 5x_1 + 2x_4 = 20
 \end{array}
 \end{array}
 \Rightarrow
 \begin{array}{l}
 \text{Matrix} \\
 \begin{bmatrix}
 0 & -5 & 1 & 1 & 5 \\
 2 & 1 & 3 & 3 & 11 \\
 1 & -2 & 2 & 2 & 8 \\
 5 & 0 & 0 & 2 & 20
 \end{bmatrix}
 \end{array}
 \Rightarrow
 \begin{array}{l}
 \text{REF} \\
 \begin{bmatrix}
 1 & -2 & 2 & 2 & 8 \\
 0 & 5 & -1 & -1 & -5 \\
 0 & 0 & -8 & -6 & -10 \\
 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \end{array}
 \Rightarrow$$

$$\begin{array}{l}
 \text{RREF} \\
 \begin{bmatrix}
 1 & 0 & 0 & 2/5 & 4 \\
 0 & 1 & 0 & -1/20 & -3/4 \\
 0 & 0 & 1 & 3/4 & 5/4 \\
 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \end{array}
 \Rightarrow
 \begin{array}{l}
 \text{System} \\
 \begin{array}{r}
 x_1 + 2/5x_4 = 4 \\
 x_2 - 1/20x_4 = -3/4 \\
 x_3 + 3/4x_4 = 5/4 \\
 0 = 0
 \end{array}
 \end{array}
 \Rightarrow
 \begin{array}{l}
 \text{Solution} \\
 \left\{ \begin{array}{l}
 x_1 = 4 - 2/5x_4 \\
 x_2 = -3/4 + 1/20x_4 \\
 x_3 = 5/4 - 3/4x_4 \\
 x_4 \text{ is free}
 \end{array} \right.
 \end{array}$$

DEF The basic (or leading) variables are the variables corresponding to the pivot columns .

EX: x_1, x_2, x_3 are basic variables

DEF Any remaining variables not associated with the pivot columns are called free variables .

EX: x_4 is a free variable

\implies Solution: