Method of Variation of Parameters

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

0. Put in standard form

$$y'' + p(x)y' + q(x)y = g(x).$$

1. Find fundamental solution set $\{y_1, y_2\}$ for the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

2. Determine $v_1(x)$ and $v_2(x)$ by

(a) Solving the system

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= g \end{aligned} \quad \text{for } v_1' \text{ and } v_2'$$

OR

Simplifying the formulas

$$v_1' = \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}$$

$$v_2' = \frac{g(x)y_1(x)}{W[y_1, y_2](x)}$$

(b) Then integrating to obtain $v_1(x)$ and $v_2(x)$.
[For simplicity, take integration constants to be 0.]

3. Substitute $v_1(x)$ and $v_2(x)$ into the form of

$$Y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

4. Form the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + Y_p(x).$$

Example:

$$y'' + y = \sec x$$

$\checkmark$

$$\begin{aligned} &y'' + y = 0 \\ &r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \\ &\qquad\qquad\qquad\quad \{\cos x, \sin x\} \end{aligned}$$

Solve: (See back)

$$\begin{aligned} \cos x \cdot v_1' + \sin x \cdot v_2' &= 0 \\ -\sin x \cdot v_1' + \cos x \cdot v_2' &= \sec x \end{aligned}$$

$$v_1 = \int -\tan x \, dx = -\ln|\sec x| + c_1 = \ln|\cos x| + c_1$$

$$v_2 = \int 1 \, dx = x + c_2$$

$$Y_p(x) = \ln|\cos x| \cdot \cos x + x \sin x$$

$$y(x) = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$