Given

PDE: $\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < L,\ t > 0$

BC: $u(0,t) = u(L,t) = 0$ $\quad t > 0$

IC: $u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = g(x)$ $\quad 0 < x < L$

We showed that the solution when $g(x) = 0$ was

$$u(x,t) = \tfrac{1}{2}\left[\,\overline{f}_o(x-ct) + \overline{f}_o(x+ct)\right]$$

- Solution only depended on the shape of the IC $f(x)$: Left and right moving waves added together (standing waves).
- Solution is in a form that is no longer an infinite sum with coefficient formula.

Let's use the quantities $x - ct$ and $x + ct$ to derive the solution in a different way.

Change of variables: Let $w = x + ct$ and $z = x - ct$. Then $u(x,t) \equiv v(w,z)$. [i.e. u and v can be interchanged.]

Use the chain rule compute the derivatives.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\cdot \underline{\quad 1 \quad} + \frac{\partial u}{\partial z}\cdot \underline{\quad 1 \quad}$$ Fill in the derivatives of w and z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial z}$$ In operator notation $\dfrac{\partial[\ ]}{\partial x} = \dfrac{\partial[\ ]}{\partial w} + \dfrac{\partial[\ ]}{\partial z} \qquad (*)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial w} + \frac{\partial v}{\partial z} \qquad (**)$$ Interchange $u \Leftrightarrow v$ on the RHS

Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right]$$

$$= \frac{\partial}{\partial w}\left[\ \frac{\partial u}{\partial x}\ \right] + \frac{\partial}{\partial z}\left[\ \frac{\partial u}{\partial x}\ \right] \qquad \text{From } (*)$$

$$= \frac{\partial}{\partial w}\left[\ \frac{\partial v}{\partial w} + \frac{\partial v}{\partial z}\ \right] + \frac{\partial}{\partial z}\left[\ \frac{\partial v}{\partial w} + \frac{\partial v}{\partial z}\ \right] \qquad \text{From } (**)$$

$$= \frac{\partial^2 v}{\partial w^2} + \frac{\partial^2 v}{\partial w \partial z} + \frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2}$$

$$= \frac{\partial^2 v}{\partial w^2} + 2\frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2} \qquad \textbf{(1)}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t} = \frac{\partial u}{\partial w}\cdot \underline{\quad c \quad} + \frac{\partial u}{\partial z}\cdot \underline{\quad -c \quad}$$

Fill in the derivatives of w and z.

$$\frac{\partial u}{\partial t} = c\left(\frac{\partial u}{\partial w} - \frac{\partial u}{\partial z}\right)$$

In operator notation $\frac{\partial[\ ]}{\partial t} = c\left(\frac{\partial[\ ]}{\partial w} - \frac{\partial[\ ]}{\partial z}\right)$ $(*)$

$$\frac{\partial u}{\partial t} = c\left(\frac{\partial v}{\partial w} - \frac{\partial v}{\partial z}\right) \qquad (**)$$

Interchange $u \Leftrightarrow v$ on the RHS

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] = c\left(\frac{\partial}{\partial w}\left[\ \frac{\partial u}{\partial t}\ \right] - \frac{\partial}{\partial z}\left[\ \frac{\partial u}{\partial t}\ \right]\right) \qquad \text{From } (*)$$

$$= c\left(\frac{\partial}{\partial w}\left[\ c\left(\frac{\partial v}{\partial w} - \frac{\partial v}{\partial z}\right)\ \right] - \frac{\partial}{\partial z}\left[\ c\left(\frac{\partial v}{\partial w} - \frac{\partial v}{\partial z}\right)\ \right]\right) \qquad \text{From } (**)$$

$$= c^2\left(\frac{\partial^2 v}{\partial w^2} - \frac{\partial^2 v}{\partial w \partial z} - \frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$= c^2\left(\frac{\partial^2 v}{\partial w^2} - 2\frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \textbf{(2)}$$

Substitute **(1)** and **(2)** into the PDE $\Rightarrow$

$$\frac{1}{c^2}\left[c^2\left(\frac{\partial^2 v}{\partial w^2} - 2\frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2}\right)\right] = \frac{\partial^2 v}{\partial w^2} + 2\frac{\partial^2 v}{\partial z \partial w} + \frac{\partial^2 v}{\partial z^2}$$

which reduces to $\frac{\partial^2 v}{\partial w \partial z} = 0$

That is, $\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial w}\right] = 0$

Integrate with respect to z: $\int \frac{\partial}{\partial z}\left[\frac{\partial v}{\partial w}\right] dz = \int 0\, dz \implies \frac{\partial v}{\partial w} = \theta(w)$ for some function θ (constant w/rt z).

Integrate with respect to w: $\int \frac{\partial v}{\partial w}\, dw = \int \theta(w)\, dw \implies v(w,z) = \int \theta(w)\, dw + \phi(z)$

[$\phi(z)$ is the integration "constant."]

i.e. $v(w,z) = \psi(w) + \phi(z)$ for arbitrary functions of w and z.

Recall that $w = \underline{\quad x + ct \quad}$ and $z = \underline{\quad x - ct \quad}$ and $v(w,z) \equiv u(x,t)$, then

$$\boxed{u(x,t) = \psi(x+ct) + \phi(x-ct)}$$

This form of the general solution to the wave equation is called ___d'Alembert's solution___ .

Again, we see that it is the superposition of 2 waves, one traveling left and one traveling right, with speed c.

But how do we find the functions ψ and ϕ?

Answer: This solution satisfies the PDE, but we haven't yet used the BC's and IC's.