The heat equation (and later the wave equation) led to an eigenvalue problem of the form:

$\phi'' + \lambda^2\phi = 0, \qquad l < x < r$

$\alpha_1\phi(l) - \alpha_2\phi'(l) = 0$

$\beta_1\phi(r) + \beta_2\phi'(r) = 0$

(a). Section 2.3 Fixed End Temperatures: $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$

$$\left.\begin{array}{ll} \phi'' + \lambda^2\phi = 0, & 0 < x < L \\ \phi(0) = 0, & \phi(L) = 0 \end{array}\right\} \Longrightarrow \text{Eigenvalues and Eigenfunctions:}$$

(b). Section 2.4 Insulated Bar: $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$

$$\left.\begin{array}{ll} \phi'' + \lambda^2\phi = 0, & 0 < x < L \\ \phi'(0) = 0, & \phi'(L) = 0 \end{array}\right\} \Longrightarrow \text{Eigenvalues and Eigenfunctions:}$$

(c). Section 2.5 Different Boundary Conditions: $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$

$$\left.\begin{array}{ll} \phi'' + \lambda^2\phi = 0, & 0 < x < L \\ \phi(0) = 0, & \phi'(L) = 0 \end{array}\right\} \Longrightarrow \text{Eigenvalues and Eigenfunctions:}$$

(d). Section 2.6 Convection: $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = h$ and $\beta_2 = -\kappa$

$$\left.\begin{array}{ll} \phi'' + \lambda^2\phi = 0, & 0 < x < L \\ \phi(0) = 0, & h\phi(L) + \kappa\phi'(L) = 0 \end{array}\right\} \Longrightarrow \text{Eigenvalues and Eigenfunctions:}$$

Cases (c) and (d) led to non-standard Fourier Series, but by direct calculation we showed that the orthogonality of eigenfunctions still held.

i.e. $\displaystyle\int_0^L \phi_n(x)\phi_m(x)\,dx =$

Show that the orthogonality condition holds for any eigenfunctions of the general eigenvalue problem given at the top of p. 1.

Let ϕ_n and ϕ_m be eigenfunctions that correspond to two different eigenvalues λ_n^2 and λ_m^2.

Then ϕ_n and ϕ_m satisfy the ODE and BC's.

i.e. (1) and (2) + BC's

Multiply (1) by ϕ_m and (2) by ϕ_n and subtract:

$$\phi_n''\phi_m + \lambda_n^2\phi_n\phi_m - (\phi_m''\phi_n + \lambda_m^2\phi_m\phi_n) = 0$$

Integrate both sides over the interval $l < x < r$:

$$
\begin{aligned}
(\lambda_m^2 - \lambda_n^2) \int_l^r \phi_m \phi_n \, dx &= \int_l^r \phi_n'' \phi_m - \phi_m'' \phi_n \, dx \\
&\overset{\text{IBP}}{=} \\
&= \phi_n' \phi_m - \phi_m' \phi_n \Big|_l^r \\
&=
\end{aligned}
$$

From BC1 at $x = l$:

$\alpha_1 \phi_m(l) - \alpha_2 \phi_m'(l) = 0$

$\alpha_1 \phi_n(l) - \alpha_2 \phi_n'(l) = 0$

From BC2 at $x = r$:

$\beta_1 \phi_m(r) + \beta_2 \phi_m'(r) = 0$

$\beta_1 \phi_n(r) + \beta_2 \phi_n'(r) = 0$

Repeat the process on p.2-3 to derive the orthogonality condition for the more generalized eigenvalue problem:

$\left[s(x)\phi'(x)\right]' - q(x)\phi(x) + \lambda^2 p(x)\phi(x) = 0, \qquad l < x < r$

$\alpha_1\phi(l) - \alpha_2\phi'(l) = 0$

$\beta_1\phi(r) + \beta_2\phi'(r) = 0$

Additional Homework Section 2.7, #1, 4