Solve the following Heat Equation problem:

$$\begin{array}{rcl} \text{PDE:} & \dfrac{1}{k}\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < L \\ \text{BCs:} & u(0,t) = T_0, \ \ \dfrac{\partial u}{\partial x}(L,t) = 0 & t > 0 \\ \text{IC:} & u(x,0) = f(x) & 0 < x < L \end{array}$$

Steady State Problem: As $t \to \infty$, $\dfrac{\partial u}{\partial t} \to$ ________ and $u(x,t) \to$ __________

ODE: $v'' = 0$ BCs: $v(0) = T_0$ and ______________

$v(x) =$ ______________ $\Rightarrow$ $v(0) =$ _____________

$v'(x) =$ ________ $\Rightarrow$ $v'(L) =$ ____________

Steady State Solution: $v(x) =$ ________

Transient Problem: $u(x,t) = w(x,t) + v(x)$

PDE: $\dfrac{1}{k}\dfrac{\partial}{\partial t}[w+v] = \dfrac{\partial^2}{\partial x^2}[w+v] \Rightarrow \dfrac{1}{k}\left(\dfrac{\partial w}{\partial t} + \underline{\qquad\quad}\right) = \underline{\qquad\qquad} + v''$ But $v'' =$ ________ $\Rightarrow \dfrac{1}{k}\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}$

BC1: $u(0,t) = w(0,t) + v(0) = T_0 \Rightarrow w(0,t) +$ ________ $= T_0 \Rightarrow$ $w(0,t) =$ ________

BC2: $\dfrac{\partial u}{\partial x}(L,t) = \dfrac{\partial}{\partial x}[w+v]\,|_{(L,t)} = \dfrac{\partial w}{\partial x}(L,t) +$ __________ $= 0 \Rightarrow$ $\dfrac{\partial w}{\partial x}(L,t) = 0$

IC: $u(x,0) = w(x,0) + v(x) = f(x) \Rightarrow$ $w(x,0) = f(x) - v(x) = f(x) -$ ________ $= g(x)$

Solve by <u>Separation of Variables</u>

Let $w(x,t) =$ _____________

PDE: $\Rightarrow \dfrac{1}{k}\phi T' = \phi'' T$ $\Rightarrow$ Divide by ϕT $\Rightarrow$ _________________ $\Rightarrow$ $\dfrac{1}{k}\dfrac{T'}{T} = \dfrac{\phi''}{\phi} = \text{constant}$

BC: $w(0,t) = \phi(0)T(t) = 0 \Rightarrow$ _____________ and $\dfrac{\partial w}{\partial x}(L,t) = \phi'(L)T(t) = 0 \Rightarrow$ _____________

Solve: $\dfrac{1}{k}\dfrac{T'}{T} = \dfrac{\phi''}{\phi} = \text{constant}$ $\qquad \phi(0) = 0, \quad \phi'(L) = 0$

Case 1: constant $= p^2 > 0$

$\dfrac{\phi''}{\phi} = p^2 \Rightarrow \qquad \phi'' - p^2\phi = 0 \quad \phi(0) = 0, \phi'(L) = 0$

$\phi(x) =$ ____________________ $\Rightarrow \phi(0) = C_1 = 0$

$\phi'(x) = C_2 p \cosh px \Rightarrow \phi'(L) =$ ____________ $= 0$, which is only possible if $C_2 = 0 \Rightarrow$ Trivial Solution

Case 2: constant $= 0$

$\phi'' = 0 \quad \phi(0) = 0, \phi'(L) = 0$

$\phi(x) =$ ____________ $\Rightarrow \phi(0) =$ ________ $= 0$

$\phi'(x) =$ ________ $\Rightarrow \phi'(L) =$ ________ $= 0 \Rightarrow$ ________________

Case 3: constant $= -\lambda^2 < 0$

$\dfrac{\phi''}{\phi} = -\lambda^2 \Rightarrow \qquad$ ______________ $\quad \phi(0) = 0, \phi'(L) = 0$

$\phi(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \Rightarrow \phi(0) =$ __________

$\phi'(x) =$ ____________ $\Rightarrow \phi'(L) = C_2 \lambda \cos \lambda L = 0$

$\Rightarrow C_2 = 0$ (trivial solution) or $\cos \lambda L = 0 \Rightarrow \lambda L =$ ____________ $\Rightarrow \lambda =$ ____________

Eigenvalues: $\lambda_n^2 = \left(\frac{(2n-1)}{2L}\pi\right)^2$ for $n = 1, 2, 3, \ldots$ (i.e. $\lambda_n = \dfrac{(2n-1)}{2L}\pi$)

Eigenfunctions: $\phi_n(x) =$ __________

Solve the associated T-eqn: $\qquad \dfrac{1}{k}\dfrac{T'}{T} = -\lambda^2 \quad \Rightarrow \quad T' + k\lambda^2 T = 0$

$\Rightarrow T(t) =$ ____________ $\qquad \Rightarrow \qquad T_n(t) = e^{-k\lambda_n^2 t}$

$\Rightarrow w_n(x,t) = \phi_n(x) T_n(t) =$ ________________

General Solution to the Transient Problem: $w(x,t) = \sum_{n=1}^{\infty}$ ____________________, where $\lambda_n = \frac{(2n-1)}{2L}\pi$

Now use the IC:

$$w(x,0) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x = \underbrace{\sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)}{2L}\pi x = g(x)}_{\text{Is this a Fourier Series for } g(x)???} \quad (*)$$

If so, find a formula for the coefficients b_n.

From previous homework, you evaluated the integrals to show the following orthogonality condition:

$$\int_0^L \sin \lambda_n x \sin \lambda_m x = \begin{cases} & n = m \\ & n \neq m \end{cases} \quad \text{where } \lambda_n = \frac{(2n-1)}{2L}\pi$$

Use the orthogonality condition to derive the coefficient formula:

Multiply (*) by $\sin \lambda_m x$ and integrate from O to L:

$$\sum_{n=1}^{\infty} b_n \sin \lambda_n x \qquad = \qquad g(x) \sin \lambda_m x$$

Interchange the sum and integral: $\sum_{n=1}^{\infty} b_n \int_0^L \sin \lambda_n x \sin \lambda_m x \, dx = \int_0^L g(x) \sin \lambda_m x \, dx$

In the sum on the LHS, each of the integrals are 0 except when $n = m \Rightarrow$:

$$b_m \cdot \int_0^L \sin \lambda_m x \sin \lambda_m x \, dx = b_m \cdot \underline{\qquad\qquad} = \int_0^L g(x) \sin \lambda_m x \, dx$$

$$b_m = \frac{2}{L} \int_0^L g(x) \sin \lambda_m x \, dx \qquad \longleftarrow \text{formula for the coefficients}$$

Full Transient Problem Solution:

$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\lambda_n^2 t} \sin \lambda_n x,$ where $b_n = \frac{2}{L} \int_0^L g(x) \sin \lambda_n x \, dx$ and $\lambda_n =$ for $n = 1, 2, 3, \ldots$

Full Solution to Original Problem:

$u(x,t) = v(x) + w(x,t)$

$$u(x,t) = \underline{\hspace{2cm}} + \sum_{n=1}^{\infty} b_n e^{-k\lambda_n^2 t} \sin \lambda_n x, \qquad \text{where } b_n = \frac{2}{L}\int_0^L g(x) \sin \lambda_n x \, dx \text{ and } \lambda_n = \frac{(2n-1)}{2L}\pi$$

Ex: Solve the original problem for the initial condition $u(x,0) = T_1$ on $0 < x < L$. [Find the coefficients b_n.]

Ex: Solve the following Heat Equation problem: [Use a separate sheet of paper]

$$\begin{aligned} &\text{PDE:} && \frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{T}{L^2} && 0 < x < L \\ &\text{BCs:} && u(0,t) = T_0, \ \frac{\partial u}{\partial x}(L,t) = 0 && t > 0 \\ &\text{IC:} && u(x,0) = T_0 && 0 < x < L \end{aligned}$$

[Hint: Will the SS Problem be the different than the previous problem? Will the Transient Problem be different than the previous problem? Will the $g(x)$ be different and thus the coefficients b_n than the last problem? Only re-solve the parts(s) that are different.]

Homework: Section 2.5, p. 162: #3, 6, 7, 9, Project 2.5(a,b)