$$\text{PDE:} \quad \frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, t > 0$$

$$\text{BC:} \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0 \qquad t > 0$$

$$\text{IC:} \quad u(x,0) = f(x) \qquad 0 < x < L$$

1. Give a physical interpretation of the boundary conditions.

Solve the following Heat Equation problem:

2. Steady-State Problem: As $t \to \infty$, $\dfrac{\partial u}{\partial t} \to$ _______ and $u(x,t) \to$ __________

ODE: $v'' = 0$ BC: $v'(0) = 0$ and ______________

$v(x) =$ ______________ $\Rightarrow$ $v'(0) =$ ____________

$v'(x) =$ ________ $\Rightarrow$ $v'(L) =$ ____________

Steady-State Solution: $v(x) =$ ________

Is the Steady State Solution Unique? _________

Use physical reasoning to find the constant steady-state temperature. [See board.]

3. <u>Transient Problem</u>: $u(x,t) = w(x,t) + v(x)$

Look at the original PDE and BC's for u. Without substituting, explain why the Transient PDE and BC's will be the same.

Verify that they are the same by substitution.

PDE: $\frac{1}{k}\frac{\partial}{\partial t}[w+v] = \frac{\partial^2}{\partial x^2}[w+v] \Rightarrow \frac{1}{k}\left(\frac{\partial w}{\partial t} + \_\_\_\_\_\_\_\_\right) = \_\_\_\_\_\_\_\_\_\_ + v''$ But $v'' = \_\_\_\_\_\_\_\_ \Rightarrow \frac{1}{k}\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}$

BC1: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial w}{\partial x}(0,t) + v''(0) = 0 \Rightarrow \frac{\partial w}{\partial x}(0,t) + \_\_\_\_\_\_\_\_ = 0 \Rightarrow$ $\frac{\partial w}{\partial x}(0,t) = \_\_\_\_\_\_\_\_$

BC2: $\frac{\partial u}{\partial x}(L,t) = \frac{\partial}{\partial x}[w+v]\,|_{(L,t)} = \frac{\partial w}{\partial x}(L,t) + \_\_\_\_\_\_\_\_\_\_ = 0 \Rightarrow$ $\frac{\partial w}{\partial x}(L,t) = 0$

IC: $u(x,0) = w(x,0) + v(x) = f(x) \Rightarrow$ $w(x,0) = f(x) - v(x) = f(x) - \_\_\_\_\_\_\_\_ = g(x)$

Solve by <u>Separation of Variables</u>

$w(x,t) = \_\_\_\_\_\_\_\_\_\_\_\_$

PDE: $\Rightarrow \frac{1}{k}\phi T = \phi'' T$ $\Rightarrow$ $\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$ $\Rightarrow$ $\frac{1}{k}\frac{T'}{T} = \frac{\phi''}{\phi} = \text{constant}$

BCs: $\phi'(0)T(t) = 0 \Rightarrow$ $\_\_\_\_\_\_\_\_\_\_\_\_$ and $\phi'(L)T(t) = 0 \Rightarrow \_\_\_\_\_\_\_\_\_\_\_\_$

Case 1: constant $= p^2 > 0$

$$\frac{\phi''}{\phi} = p^2 \Rightarrow \qquad \phi'' - p^2\phi = 0 \quad \phi'(0) = 0, \phi'(L) = 0$$

$\phi(x) =$ ____________________________

$\phi'(x) = C_1 p \sinh px + C_2 p \cosh px \Rightarrow \phi'(0) =$ ________________ $= 0 \Rightarrow C_2 p =$ ________ $\Rightarrow C_2 = 0$

$\phi'(x) = C_1 p \sinh px \Rightarrow \phi'(L) =$ _________________ $= 0$
$C_1 = 0 \Rightarrow$ Trivial Solution OR $\sinh pL = 0 \Rightarrow$ Not possible for $p \neq 0$

$\Rightarrow$ Trivial Solution

Case 2: constant $= 0$

$\phi'' = 0 \quad \phi'(0) = 0, \phi'(L) = 0$

$\phi(x) =$ _______________

$\phi'(x) =$ _________ $\Rightarrow \phi'(0) =$ _________ $= 0$

$\phi'(x) =$ _________ $\Rightarrow \phi'(L) =$ _________ $= 0 \Rightarrow \phi =$ _________ (Arbitrary Constant)

Case 3: constant $= -\lambda^2 < 0$

Finish Case 3.

Summarize the eigenvalues and eigenfunctions.

Write the final solution.

4. Find the limit as $t \to \infty$ for your solution $u(x,t)$. How does this limit relate to the steady-state solution?

5. Suppose the initial condition is given by $u(x,0) = T_0 + \dfrac{T_1 - T_0}{L}x$.

(a). Sketch a picture of this initial condition (assume $T_1 > T_0$).

(b). Solve for the coefficients with this initial condition.

(c). Write down the final solution for $u(x,t)$.

Homework: Section 2.4: #2, 3, 5, 8

Verify the following orthogonality conditions by evaluating the integral

$$\int_0^L \sin\left(\frac{(2n-1)\pi x}{2L}\right)\sin\left(\frac{(2m-1)\pi x}{2L}\right)\,dx$$

for

(a). $n = m$

(b). $n \neq m$