1. For $f(x) = x$ on $[0, L]$, the Fourier Sine Series is given by

$$x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{L}x\right) \tag{1}$$

(a). Sketch a picture of the periodic extension for this series.

(b). Differentiate both sides of (1).

(**i**) Does the RHS look like a Fourier Sine or Cosine Series? If so, which one?

(**ii**) If so, what are the coefficients are for the Fourier (Sine or Cosine) Series of the function $g(x) = 1$.

$A_0 =$ 0 and $A_n =$ $2(-1)^{n+1}$, $n = 1, 2, 3, \ldots$

(c). Find the Fourier Cosine Series for $g(x) = 1$ using the integral formulas for the coefficients.

(**i**) Are the results of **(b)** and **(c)** the same?

[Note: From part(c) we see that the Fourier Cosine Series for $g(x) = 1$ is itself $g(x) = 1$.]

Based on this problem, do you think that you can you find the Fourier Series of $f'(x)$ by differentiating the Fourier Series of $f(x)$?

2. For $f(x) = x$ on $[0, L]$, the Fourier Cosine Series is given by

$$x = L + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi}{L} x\right) \tag{2}$$

(a). Sketch a picture of the periodic extension for this series.

(b). Differentiate both sides of (2).

(i) Does the RHS look like a Fourier Sine or Cosine Series? If so, which one?

(ii) If so, what are the coefficients are for the Fourier (Sine or Cosine) Series of the function $g(x) = 1$.

$$B_n = -\frac{2}{\pi}\left(\frac{(-1)^n - 1}{n}\right), \qquad n = 1, 2, 3, \ldots$$

(c). Find the Fourier <u>Sine</u> Series for $g(x) = 1$ using the integral formulas for the coefficients.

(i) Are the results of **(b)** and **(c)** the same?

(ii) Hmm... why did it work this time?

[Next page.]

3. Look at the periodic extension you sketched in **(1a)** and **(2a)**. What significant difference (besides odd vs. even) do you notice about the two different extensions?

Use this observation to fill in the blank in the following Theorem:

<u>THEOREM</u> If $f(x)$ is periodic, ___continuous___ , and piecewise smooth, then the differentiated Fourier Series of $f(x)$ converges to $f'(x)$ at every point x where $f''(x)$ exists.

i.e. If $f(x)$ satisfies those conditions and

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

then

$$f'(x) = \sum_{n=1}^{\infty} \left(-\frac{n\pi}{L} a_n \sin\left(\frac{n\pi}{L}x\right) + \frac{n\pi}{L} b_n \cos\left(\frac{n\pi}{L}x\right)\right)$$

or

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

then

$$f'(x) = \sum_{n=1}^{\infty} (-na_n \sin(nx) + nb_n \cos(nx))$$

4. Look up Theorem 3 on p. 78 and fill in the blank below:

<u>THEOREM</u> If $f(x)$ is ___periodic and piecewise continuous___ then the Fourier Series of $f(x)$ may be integrated term by term and will equal the integral of $f(x)$.

i.e. $\int_a^b f(x)\, dx =$

How do the requirements for $f(x)$ differ for integration versus differentiation? Which one has stricter requirements?

5. The Fourier Sine Series for $f(x) = 1$ on $[0, L]$ is given by

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi}{L} x\right) \tag{3}$$

(a). Change the variable to t and then integrate both sides of (3) with respect to t from 0 to x.

You should get a sum of terms involving cosines of "x-stuff" and a separate sum involving just constants. You may need to distribute and split up the sum.

(b). The result of part **(a)** is the Fourier Cosine Series for which function?

(c). What does the general form for a Fourier **<u>Cosine</u>** Series look like?

(d). Compare **(c)** with your result in part **(a)**. What does the coefficient A_0 equal? i.e. What is the constant term in **(a)**?

(e). Use the function from part **(b)** and the integral formula to find the coeffcient A_0.

(f). Since both parts **(d)** and **(e)** give valid answers for the coefficient A_0, they should be equal. Set them equal and solve for the summation only.

(g). Woo Hoo!! We just found the sum of what series? Write it out long-hand, too.

6. Your result of integrating the Fourier Series for $f(x) = 1$ should have given you: [Using the simpler version of A_0.]

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)\pi}{L} x\right)$$

(a). Integrate both sides again with respect to x [You can take the integration constant(s) to be 0].

(b). Is the result a Fourier Series? Why or why not?

(c). What is the Fourier Sine Series for $h(x) = \frac{1}{2}x^2 - \frac{L}{2}x$?

Homework:

- Read Section 1.5. For each theorem 1-6, state the theorem and then explain it in your own words.
- Section 1.5, p. 82: #1, 2, 5, 9, 10