1. Recall, Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$ $e^{-i\theta} = \cos\theta - i\sin\theta$

(a). Use Euler's Formulas above to verify the following definitions:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

(b). Use the definitions from part (a) to write $\cos(nx)$ and $\sin(nx)$.

2. The Fourier Series for a $2\pi-$periodic function is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Use the definitions from #1 to rewrite the Fourier Series using e^{inx} and e^{-inx}. Then distribute and collect all the e^{inx} and e^{-inx} terms.

3.

(a). What is the coefficient of the e^{inx} term? [Rationalize it, if possible.]

(b). What is the coefficient of the e^{-inx} term? [Rationalize it, if possible.]

(c). How are these two coefficients related? i.e. What do we call them?

4. Let
$$\begin{aligned} c_0 &= a_0 \\ c_n &= \text{coefficient of } e^{inx} \\ \overline{c_n} &= \text{coefficient of } e^{-inx} \end{aligned}$$
$\overline{c_n}$ denotes complex conjugate

Then if you rewrite the Fourier Series in terms using complex exponentials and these coefficients, you should get

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \overline{c_n}\, e^{-inx}$$

Clearly define c_0, c_n, and $\overline{c_n}$ in terms of a_n and b_n. [See #2 and #3(a,b).]

5. Let $\overline{c_n}$ be denoted by c_{-n} since it corresponds to the coefficients preceding e^{-inx} (i.e. term involving $-n$).
Now
$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + c_{-n}\, e^{-inx}$$

which can be condensed into the single term series below
[Fill in the bounds – it may be helpful to write out a few of the terms in the series.]

$$f(x) = \sum_{n=}^{} \; c_n e^{inx}$$

This form is called the Complex Fourier Series for $f(x)$.

6. Recall the Fourier coefficients for a $2\pi-$periodic function are given by the integrals

$a_n =$ and $b_n =$ for $n = 1, 2, \ldots$

(a). Use the relationship $c_n = \frac{1}{2}a_n - \frac{1}{2}ib_n$, the integrals above, and the complex form of the sine and cosine functions to derive the formula for the complex Fourier coefficients c_n for $n = 1, 2, \ldots$.

(b). In part (a), you should have gotten $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. Verify that this formula will work for c_0.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \qquad \text{for } all \text{ integers } n$$ (e.g. negative, positive, and zero).

7. Find the Complex Fourier Coefficients for $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$

8. Given $f(x) = e^{\alpha x}$ on $-\pi < x < \pi$.

(a). Find the Complex Fourier Coefficients for

(b). Find a_n and b_n from part (a) using the relationship $c_n = \frac{1}{2}a_n - i\frac{1}{2}b_n$ [Simplify.]

Homework Section 1.10, p. 104, #3, 4(b), 6