Details for Solving the Logistic Model

(variation of these details are in Section 2.5)

Solve the Logistic Model for Constrained Population Growth.

$$\frac{dP}{dt} = -aP(P-K) \quad ; \quad P(0) = P_0 \qquad \left(K = \frac{r}{a}\right)$$

Separable: $\frac{1}{P(P-K)}\,dP = -a\,dt$

$$\int \frac{1}{P(P-K)}\,dP = \int -a\,dt \qquad (*)$$

Partial Fraction Decomposition:

$(**)$ $\frac{1}{P(P-K)} = \frac{A}{P} + \frac{B}{P-K}$ ← Decompose into 2 fractions w/ unknown A & B

$= \frac{A(P-K)+BP}{P(P-K)}$ ← Recombine w/ LCD

$\Rightarrow\ 1 = AP - AK + BP$ ← Compare numerators (since denom are the same)

$1 = (A+B)P - AK$

LHS = RHS requires

$P:\ 0 = A+B$ and const: $1 = -AK$ } 2 eqns for 2 unknowns A & B $\Rightarrow A = -\frac{1}{K} \Rightarrow A+B=0$, $-\frac{1}{K}+B=0 \Rightarrow B = \frac{1}{K}$

So $(**)$ becomes $\frac{1}{P(P-K)} = \frac{-1/K}{P} + \frac{1/K}{P-K} = -\frac{1}{K}\left(\frac{1}{P} - \frac{1}{P-K}\right)$

Substitute into $(*)$

$$\int -\frac{1}{K}\left(\frac{1}{P} - \frac{1}{P-K}\right)dP = -a\,dt$$

$$-\frac{1}{K}\left(\ln|P| - \ln|P-K|\right) = -at + C$$

$$\ln|P| - \ln|P-K| = aKt + C_2 \quad , (C_2 = -KC)$$

$$\ln\left|\frac{P}{P-K}\right| = aKt + C_2$$

$$e^{\ln\left|\frac{P}{P-K}\right|} = e^{aKt + C_2}$$

$$\left|\frac{P}{P-K}\right| = e^{aKt}\cdot e^{C_2}$$

$$\left|\frac{P}{P-K}\right| = C_3\, e^{aKt}$$

$\frac{P}{P-K} = C_3\, e^{aKt}$ ← C_3 can "absorb" any + or −

Use IC: $P(0) = P_0$: $\frac{P_0}{P_0 - K} = C_3 e^0 \Rightarrow C_3 = \frac{P_0}{P_0 - K}$

$\Rightarrow \frac{P}{P-K} = \frac{P_0}{P_0 - K} e^{aKt}$ ← Still implicit. Continue to get $P = P(t)$ explicitly

$$\frac{P}{P-K} = \frac{P_0}{P_0-K}e^{akt}$$ Cross-Multiply

$$P(P_0-K) = P_0(P-K)e^{akt}$$

$$PP_0 - PK = PP_0e^{akt} - P_0Ke^{akt}$$

$$PP_0 - PK - PP_0e^{akt} = -P_0Ke^{akt}$$

$$P(P_0 - K - P_0e^{akt}) = -P_0Ke^{akt}$$

$$P(t) = \frac{-P_0Ke^{akt}}{P_0-K-P_0e^{akt}}$$

Mult by $\frac{-e^{-akt}}{-e^{-akt}} = 1$

$$= \frac{-P_0Ke^{akt}}{(P_0-K-P_0e^{akt})} \cdot \frac{-e^{-akt}}{(-e^{-akt})}$$

$$= \frac{P_0K}{-P_0e^{-akt} + Ke^{-akt} + P_0}$$

$$= \frac{P_0K}{P_0 + (K-P_0)e^{-akt}}$$

ie

$$\boxed{P(t) = \frac{P_0K}{P_0+(K-P_0)e^{-akt}} \quad \text{gives the population at time } t}$$

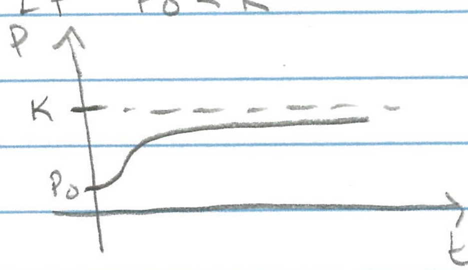
Logistic Curve or Function

Observe: $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0K}{P_0+(K-P_0)\underbrace{e^{-akt}}_{\to 0}} = \frac{\cancel{P_0}K}{\cancel{P_0}} = K$

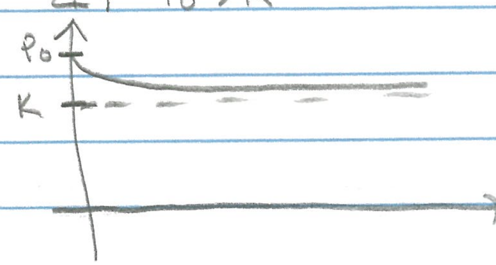
Carrying capacity

Graphs:

If $P_0 < K$



If $P_0 > K$



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Note: K and a need to be determined in some way - usually using observed - data

For example,
Use Population

1790	3.93 Million	← Let $t=0$ be 1790
1840	17.07 Million.	⇒ $t=50$
1890	62.95 Million	⇒ $t=100$

So $P(0)=3.93$ $P(50)=17.07$ $P(100)=62.95$

$$\Rightarrow P(t)=\frac{3.93}{3.93+(K-3.93)e^{-aKt}}$$ ← Still need a & K

$$P(50)=\frac{3.93}{3.93+(K-3.93)e^{-aK(50)}}=17.07$$

$$P(100)=\frac{3.93}{3.93+(K-3.93)e^{-aK(100)}}=62.95$$

2 nonlinear eqns for 2 unknowns a & K

Use Maple to Solve

$a \approx .0001214$
$K = 250.992$

$$P(t)=\frac{3.93}{3.93+(250.992-3.93)\,e^{-(.0001214)(250.992)t}}$$

$$\boxed{P(t)=\frac{3.93}{3.93+247.062\,e^{-.03047t}}}$$