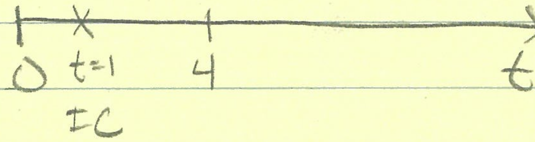
Key

1. $(t-4)y' + \ln(t)y = 2t^2+1$, $y(1)=8$

$$y' + \underbrace{\frac{\ln(t)}{t-4}}_{t>0,\ t\neq 4} y = \underbrace{\frac{2t^2+1}{t-4}}_{t\neq 4}$$



Interval: $\boxed{0<t<4}$

2. $\left.\begin{aligned} \phi(t) &= e^{rt} \\ \phi'(t) &= re^{rt} \\ \phi''(t) &= r^2e^{rt} \end{aligned}\right\} \to$

$3y'' + 4y' - 4y = 0$

$3r^2e^{rt} + 4re^{rt} - 4e^{rt} = 0$

$e^{rt}(3r^2+4r-4) = 0$

$\cancel{e^{rt} = 0}$ or $3r^2+4r-4=0$

$(3r-2)(r+2)=0$

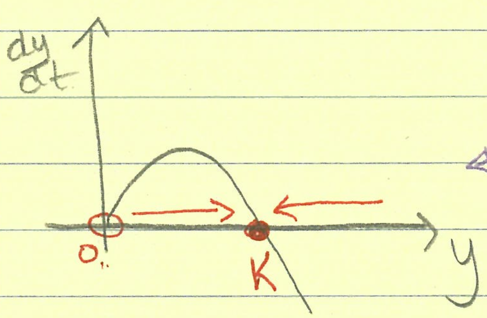
$\Rightarrow \boxed{r = \frac{2}{3}, -2}$

3. $\frac{dy}{dt} = ry\ln\left(\frac{K}{y}\right)$, ~~$r, K > 0$~~

Changed problem after key was written: r = 2, K = 80

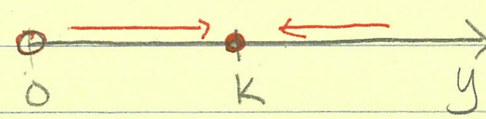
~~* Use Calculator & Choose r, K > 0~~

a)



Phase Line in Red

or



b) $\frac{dy}{dt} = 0 \Rightarrow ry\ln\left(\frac{K}{y}\right) = 0 \Rightarrow ry = 0$ or $\ln\left(\frac{K}{y}\right) = 0$

$y = 0$ or $\frac{K}{y} = 1$

$\Rightarrow y = K$

Equil: $\boxed{\begin{aligned} &y = 0 \text{ Unstable} \\ &y = K \text{ Stable} \end{aligned}}$

c) If $y_0 = \cancel{K+1}$ 100, then $\boxed{y \to K \text{ as } t \to \infty.}$

4.a) $\frac{dy}{dx} = \frac{1}{x - xy} \qquad y(1) = 3$

$= \frac{1}{x(1-y)}$

$(1-y)\,dy = \frac{1}{x}\,dx$

$\int 1-y \, dy = \int \frac{1}{x}\,dx$

$y - \frac{1}{2}y^2 = \ln|x| + C$

IC: $3 - \frac{1}{2}(3)^2 = \underbrace{\ln|1|}_{0} + C \Rightarrow C = 3 - \frac{9}{2} = \frac{-3}{2}$

$$\boxed{y - \frac{1}{2}y^2 = \ln|x| - \frac{3}{2}}$$

b) $y' + \frac{1}{1+t}y = 2t \qquad \mu = e^{\int \frac{1}{1+t}dt} = e^{\ln|1+t|}$

$= 1+t$, Note $t > -1$ [illegible] given

$\underbrace{(1+t)y' + y}_{} = 2t(1+t)$

$\frac{d}{dt}\left[(1+t)y\right] = 2t + 2t^2$

$(1+t)y = t^2 + \frac{2}{3}t^3 + C$

$$\boxed{y = \frac{t^2 + \frac{2}{3}t^3 + C}{1+t}}$$

5. $\frac{dy}{dx} = -\frac{ax+by}{bx+cy} \Rightarrow (bx+cy)\,dy + (ax+by)\,dx = 0$

$N_x = b \qquad M_y = b$ ✓

$\frac{\partial \Psi}{\partial x} = ax + by \leftarrow M$

$\Psi(x,y) = \int ax + by \, dx$

$= \frac{1}{2}ax^2 + bxy + g(y) \leftarrow (*)$

$\frac{\partial \Psi}{\partial y} = bx + g'(y) = bx + cy \leftarrow N$

$\Rightarrow g'(y) = cy \Rightarrow g(y) = \frac{1}{2}cy^2$

Sol^n (D is arbitrary const

$\Psi(x,y) = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2 \Rightarrow \boxed{\frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2 =}$

(Got cut off when scanning) $= D$

6(a) $r_i = 4$ L/min $\qquad r_o = 4$ L/min

$c_i = 2$ kg/L $\qquad c_o = \frac{Q(t)}{V(t)} = \frac{Q(t)}{500}$ kg/L

$$\frac{dQ}{dt} = r_i c_i - r_o c_o = 4\cdot 2 - 4\cdot\frac{Q(t)}{500} \Rightarrow \boxed{\frac{dQ}{dt} + \frac{1}{125}Q = 8 \quad Q(0) = 5} \text{ IVP}$$

Solve: $\mu = e^{\int \frac{1}{125}dt} = e^{\frac{1}{125}t}$ $\qquad$ Note: $\frac{1}{125} = .008$

$$\underbrace{e^{.008t}\frac{dQ}{dt} + .008e^{.008t}}Q = 8e^{.008t}$$

$$\frac{d}{dt}\left[e^{.008t}Q\right] = 8e^{.008t}$$

$$e^{.008t}Q = \int 8e^{.008t}\,dt = 8\cdot\frac{1}{.008}e^{.008t} + C$$

$$e^{.008t}Q = 1000e^{.008t} + C$$

$$Q(t) = 1000 + Ce^{-.008t}$$

IC: $\quad Q(0) = 1000 + C = 5 \Rightarrow C = -995$

$$\boxed{Q(t) = 1000 - 995e^{-.008t}}$$

(b) $Q(10) = 1000 - 995e^{-.008(10)} \approx \boxed{81.50 \text{ kg}}$

$C(10) = \frac{Q(10)}{V(10)} = \frac{81.50}{500} = \boxed{0.163 \text{ kg/L}}$

(c) $V(t) = 500 - 0.1t$ $\qquad Q(0) = 81.50$ ← reset time

$$\frac{dQ}{dt} = 4\cdot 2 - (4.1)\frac{Q(t)}{500 - 0.1t}$$

$$\boxed{\frac{dQ}{dt} + \frac{4.1}{500-0.1t}Q = 8, \quad Q(0) = 81.50}$$

7. $m = 70$ kg $\quad g = 10\ m/s^2 \quad b = 3.2 \quad v_0 = 0$

$x_0 = 0$

(a) $v(t) = \frac{mg}{b} + \left(v_0 - \frac{mg}{b}\right)e^{-\frac{b}{m}t}$

$$v(t) = \frac{(70)(10)}{3.2} + \left(0 - \frac{70(10)}{3.2}\right)e^{-\frac{3.2}{70}t}$$

$$= 218.75 - 218.75e^{-.0457t}$$

$$v_T = \lim_{t\to\infty} v(t) = \lim_{t\to\infty} 218.75 - \cancel{218.75e^{-.0457t}} \to 0$$

$$= \boxed{218.75\ m/s}$$

(b) $x(t) = \frac{mg}{b}t + \frac{m}{b}\left(v_0 - \frac{mg}{b}\right)\left(1 - e^{-\frac{b}{m}t}\right) + x_0$

$$= 218.75t + \frac{70}{3.2}(0 - 218.75)\left(1 - e^{-.0457t}\right) + 0$$

$$\boxed{x(t) = 218.75t - 4785.15625\left(1 - e^{-.0457t}\right)}$$

(c) As $t \to \infty$, the exponential term $\to 0$, so

$$\boxed{x(t) \to 218.75t - 4785.15625 \quad \text{as } t \to \infty}$$

(d) $218.75t - 4785.15625 = 5000$

$$218.75t = 9785.15625$$

$$t = \frac{9785.15625}{218.75} \approx \boxed{44.73 \text{ seconds}}$$

Note: Solving the full $x(t) = 5000$ gives $t \approx 41.44$