Exam 2 Key

1. $(t-3)y'' + \ln|t|\, y' = t \qquad y(1)=1 \quad y'(1)=-4$

$$\frac{y''}{t-3} + \frac{\ln|t|}{t-3} y' = \frac{t}{t-3}$$

discontinuities at $t=0, 3$

IC @ $t=1 \Rightarrow$ Interval: $\boxed{0<t<3}$

2. $D(D-2)^2(D^2+1)[y] = t^2 + te^{2t} + e^t \sin t$

$r = 0, 2, 2, \pm i$

$\{1, e^{2t}, te^{2t}, \cos t, \sin t\}$

$$\boxed{Y_p = t(At^2+Bt+C) + t^2(Dt+E)e^{2t} + Fe^t\cos t + Ge^t \sin t}$$

3. (a) $y'' - 4y' + 7y = 0$

$r^2 - 4r + 7 = 0$

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-28}}{2}$$

$$= \frac{4 \pm \sqrt{-12}}{2} = \frac{4 \pm 2\sqrt{3}\, i}{2}$$

$$= 2 \pm \sqrt{3}\, i$$

$$\boxed{y(t) = C_1 e^{2t}\cos\sqrt{3}t + C_2 e^{2t}\sin\sqrt{3}t}$$

(b) $x^2y'' - 5xy' + 9y = 0$

$r(r-1) - 5r + 9 = 0$

$r^2 - r - 5r + 9 = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r = 3, 3$

$$\boxed{y(x) = C_1 x^3 + C_2 x^3 \ln x}$$

4. $y''+y'-6y = 3t+10e^{-3t}$ $\quad y(0)=0,\ y'(0)=1$

$r^2+r-6=0$

$(r+3)(r-2)=0$

$r=-3,2$

$\{e^{-3t}, e^{2t}\}$

$y_p = At+B+Cte^{-3t}$

$y_p' = A -3Cte^{-3t}+Ce^{-3t}$

$y_p'' = 9Cte^{-3t}-3Ce^{-3t}-3Ce^{-3t}$

$= 9Cte^{-3t}-6Ce^{-3t}$

Subs.

$\cancel{9Cte^{-3t}}-6Ce^{-3t}+A-\cancel{3Cte^{-3t}}+Ce^{-3t}\underbrace{-6(At+B+Cte^{-3t})}_{-6At-6B-\cancel{6Cte^{-3t}}}=3t+10e^{-3t}$

$-5Ce^{-3t}-6At+(A-6B) = 3t+10e^{-3t}$

LHS = RHS

$e^{-3t}: \quad -5C=10 \Rightarrow C=-2$

$t: \quad -6A=3 \Rightarrow A=\frac{-3}{6}=-\frac{1}{2}$

Const: $\quad A-6B=0 \Rightarrow -\frac{1}{2}-6B=0$

$6B=-\frac{1}{2} \Rightarrow B=-\frac{1}{12}$

Gen sol^n

$y(t)=C_1e^{-3t}+C_2e^{2t}-\frac{1}{2}t-\frac{1}{12}-2te^{-3t}$

$y'(t)=-3C_1e^{-3t}+2C_2e^{2t}-\frac{1}{2}-2t(-3e^{-3t})+e^{-3t}(-2)$

$y(0)=C_1+C_2-\frac{1}{12}=0 \Rightarrow C_1+C_2=\frac{1}{12}$ ①

$y'(0)=-3C_1+2C_2-\frac{1}{2}-2=1 \Rightarrow -3C_1+2C_2=1+\frac{1}{2}+2=\frac{7}{2}$ ②

① x 3: $\quad 3C_1+3C_2=\frac{1}{4}$

Add $\quad 5C_2=\frac{15}{4}$

$C_2=\frac{3}{4}$

Back subs. $\quad C_1+\frac{3}{4}=\frac{1}{12} \Rightarrow C_1=\frac{1}{12}-\frac{9}{12}=\frac{-8}{12}=\frac{-2}{3}$

$$\boxed{y(t)=-\frac{2}{3}e^{-3t}+\frac{3}{4}e^{2t}-\frac{1}{2}t-\frac{1}{12}-2te^{-3t}}$$

5 $y'' + 4y' + 4y = 0 \qquad y(0) = 1 \qquad y'(0) = b$

(a) $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \qquad y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

$r = -2, -2 \qquad y'(t) = -2c_1 e^{-2t} + -2c_2 t e^{-2t} + c_2 e^{-2t}$

$y(0) = c_1 = 1$

$y'(0) = -2c_1 + c_2 = b \Rightarrow -2 + c_2 = b \Rightarrow c_2 = b+2$

$$\boxed{y(t) = e^{-2t} + (b+2) t e^{-2t}}$$

(b) $y'(t) = -2e^{-2t} - 2(b+2) t e^{-2t} + (b+2) e^{-2t} = 0$

$\underbrace{e^{-2t}}_{\text{never } 0} (-2 - 2(b+2)t + b + 2) = 0$

$\Rightarrow \cancel{-2} - 2(b+2)t + b + \cancel{2} = 0$

$-2(b+2)t + b = 0$

$2(b+2)t = b$

$$t_m \rightarrow \boxed{t = \frac{b}{2(b+2)}}$$

6 $y'' + 9y = \sec(3t)$

$y_1(t) = \cos 3t \quad y_2(t) = \sin 3t$

$$W = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3\cos^2 t + 3\sin^2 t$$
$$= 3(\cos^2 t + \sin^2 t)$$
$$= 3$$

$$v_1' = \frac{-g y_2}{W} = -\frac{\sec(3t)\cdot\sin(3t)}{3}$$

$$\Rightarrow v_1 = -\int \frac{1}{3}\frac{\sin(3t)}{\cos(3t)}\,dt = -\int \frac{1}{3}\tan 3t\,dt = \frac{-1}{3}\cdot\frac{1}{3}\ln|\sec 3t| + \underset{0}{c}$$
$$= -\frac{1}{9}\ln|\sec 3t|$$

$$v_2' = \frac{g y_1}{W} = \frac{\sec 3t \cdot \cos 3t}{3} = \frac{1}{3}$$

$$v_2 = \int \frac{1}{3}\,dt = \frac{1}{3}t + \underset{0}{c}$$

$$y_p = -\frac{1}{9}\ln|\sec 3t|\cdot\cos(3t) + \frac{1}{3}t\sin 3t$$

$$\boxed{y(t) = c_1\cos 3t + c_2\sin 3t - \frac{1}{9}\ln|\sec 3t|\cdot\cos(3t) + \frac{1}{3}t\sin 3t}$$

7. (a) $m = 12$ g $L = 4$ cm $\gamma = 360$ $g = 980$

still need k: $mg = kL$

$12(980) = k(4)$

$\Rightarrow k = \frac{\cancel{12}^{3}(980)}{\cancel{4}} = 2940$

$$\boxed{12u'' + 360u' + 2940u = 0 \qquad \begin{array}{ll} u(0) = -2 & \text{cm} \\ u'(0) = 0 & \text{cm/s} \end{array}}$$

Divide by 12

(b) $u'' + 30u' + 245u = 0$

$r^2 + 30r + 245 = 0$

$r = \frac{-30 \pm \sqrt{(-30)^2 + 4(1)(245)}}{2(1)}$

$80 = 16 \cdot 5$

$= \frac{-30 \pm \sqrt{-80}}{2} = \frac{-30 \pm 4\sqrt{5}i}{2} = -15 \pm 2\sqrt{5}i$

$$\boxed{y(t) = C_1 e^{-15t} \cos 2\sqrt{5}t + C_2 e^{-15t} \sin 2\sqrt{5}t}$$

(c) $T_d = \frac{2\pi}{\omega} = \frac{2\pi}{2\sqrt{5}} = \boxed{\frac{\pi}{\sqrt{5}}}$