Axiom 5 (The Protractor Postulate)

For every angle $\angle BAC$ there is a real number $\mu = \mu(\angle BAC)$ called the ____________________ such that

1. $0^\circ \leq \mu^\circ < 180^\circ$

2. $\mu = 0^\circ$ iff $\overrightarrow{AB} = \overrightarrow{AC}$

3. For each real number r where $0^\circ < r^\circ < 180^\circ$ and for each of the two half-planes determined by $\overleftrightarrow{AB}$, there exists a unique ray $\overrightarrow{AE}$ such that E is in the half-plane and $\mu(\angle BAE) = r^\circ$

(____________________________)

4. If the ray $\overrightarrow{AD}$ is between the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\mu(\angle BAD) + \mu(\angle DAC) = \mu(\angle BAC)$

(_________________________)

Def Two angles are **congruent** if they have the ____________________ .

i.e. $\angle BAC \cong \angle EDF$ if _______________________

Def
An angle with measure $\mu = 90°$ is a _________ angle.

An angle with measure $\mu < 90°$ is a _________ angle.

An angle with measure $\mu > 90°$ is a __________ angle.

Lemma If A, B, C, and D are four distinct points such that C and D are on the same side of $\overleftrightarrow{AB}$ and D is not on $\overleftrightarrow{AC}$, then either C is in the interior of $\angle BAD$ or D is in the interior of $\angle BAC$.

Proof Let A, B, C and D be defined as above.

Suppose D is not in the interior of $\angle BAC$ [Show ______________________________]

Then D and B are on _______________ of $\overleftrightarrow{AC}$

Therefore $\overline{BD} \cap \overleftrightarrow{AC}$______$\emptyset$ by the Plane Separation Postulate.

Let P be this __________ point of intersection.

Then P is between B and D and by theorem 3.3.10 (proved on previous worksheet), ________________ .

Therefore, P is in the interior of $\angle BAD$. [Still need to show that ______ is in the interior of $\angle BAD$.]

[How?: ____________________]

P lies on $\overleftrightarrow{AC}$ since it is the intersection point.

Therefore $\overrightarrow{AP}$ will equal $\overrightarrow{AC}$ as long as they are not ___________ rays.

Since P is in the interior of $\angle BAD$, P and D are on ____________ side of $\overleftrightarrow{AB}$.

Then P and C are also on the same side of $\overleftrightarrow{AB}$ and hence, $\overrightarrow{AP}$ and $\overrightarrow{AC}$ __________ be opposite rays.

Therefore $\overrightarrow{AP} = \overrightarrow{AC}$ $\Rightarrow$ $\overrightarrow{AB} *$ $* \overrightarrow{AD}$ $\Rightarrow$ C is in the interior of $\angle BAD$. ■

Why don't we have to prove that "If C is not in the interior of $\angle BAD$, then D is in the interior of BAC?"

<u>Theorem</u> (Betweenness Theorem for Rays). Let A, B, C, and D be four distinct points such that C and D lie on the same side of $\overleftrightarrow{AB}$. Then $\mu(\angle BAD) < \mu(\angle BAC)$ iff $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

<u>Proof</u> Let A, B, C, and D be defined as stated above.

$\Leftarrow$: Let $\overrightarrow{AD}$ be between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [Show $\mu(\angle BAD) < \mu(\angle BAC)$]

Then ________________________________ by the Protractor Postulate Part 4.

Since $\mu(DAC) > 0 \Rightarrow \mu(\angle BAD) < \mu(\angle BAC)$ by the Protractor Postulate ______________ .

$\Rightarrow$: Let $\mu(\angle BAD) < \mu(\angle BAC)$ [Show $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.]

BWOC, suppose that $\overrightarrow{AD}$ ________________ $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

<u>Case 1</u>: D lies on $\overrightarrow{AC}$.

Then $\mu(\angle BAD)$ _______ $\mu(\angle BAC)$. (*)

<u>Case 2</u>: D does not lie on $\overrightarrow{AC}$.

Then by the previous Lemma, _______ is in the interior of ___________ .

Thus, $\overrightarrow{AC}$ is between _______________ by definition of between for rays (Def 3.3.8).

Then by the first half of this theorem, _______________________ (**)

Combining the results of Case 1 and 2, we have that $\mu(\angle BAD)$_______$\mu(\angle BAC)$ $\rightarrow\leftarrow$

Therefore, $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

<u>Def</u> Let A, B, and C be three noncollinear points. A ray $\overrightarrow{AD}$ is an ________________ of $\angle BAC$ if D is in the interior of BAC and $\mu(\angle BAD) = \mu(\angle DAC)$.

[Sketch]

<u>Theorem</u> If A, B, and C are three noncollinear points, then there exists a unique angle bisector for $\angle BAC$.

<u>Proof</u> Let A, B, and C be three noncollinear points.

By the Protractor Postulate Part 1, $0° \leq \mu(\angle BAC) < 180°$. Thus, _______ $\leq \frac{1}{2}\mu(\angle BAC) <$ ________.

Then by the Protractor Postulate Part 3, there exists a unique ray $\overrightarrow{AE}$ such that $\mu(\angle BAE) =$ ______________ and E can be chosen to be on the same side of $\overleftrightarrow{AB}$ as C.

Since $\mu(\angle BAE) = \frac{1}{2}\mu(\angle BAC) < \mu(\angle BAC) \Rightarrow$ ________ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ by the Betweenness Theorem for Rays.

[continued on next page]

Then $\mu(\angle BAE) + \mu(\angle EAC) = \mu(\angle BAC)$ by ______________________________

$\Rightarrow \frac{1}{2}\mu(\angle BAC) + \mu(\angle EAC) = \mu(\angle BAC) \Rightarrow \mu(\angle EAC) = \frac{1}{2}\mu(\angle BAC) \Rightarrow \mu(\angle BAE) = \mu(\angle EAC) = \frac{1}{2}\mu(\angle BAC)$.

Therefore an $\overrightarrow{AE}$ is an angle bisector for $\angle BAC$.

[Still need to show uniqueness. – do later as homework.]

<u>Def</u> Two lines l and m are ________________ if there exists a point A that lies on both l and m and there exist points $B \in l$ and $C \in m$ such that $\angle BAC$ is a right angle.

Denoted:

[Sketch]

<u>Def</u> A ______________________ of $\overline{AB}$ is a line l such that the midpoint of $\overline{AB}$ lies on l and $\overleftrightarrow{AB} \perp l$.

[Sketch]

<u>Def</u> Two angles $\angle BAD$ and $\angle DAC$ form a ______________ if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.

[Sketch]

<u>Def</u> Two angles $\angle BAC$ and $\angle DEF$ are _________________ if $\mu(\angle BAC) + \mu(\angle DEF) = 180°$.

[Sketch]

<u>Def</u> Angles $\angle BAC$ and $\angle DAE$ form a _______________ (or are _________________) if $\overrightarrow{AB}$ and $\overrightarrow{AE}$ are opposite rays and $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are opposite rays OR if $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite rays and $\overrightarrow{AC}$ and $\overrightarrow{AE}$ are opposite rays.

[Sketch]

Homework Section 3.4, p. 55: #1[Finish Uniqueness] Section 3.5, p. 62: #2