[This worksheet assumes that you know the basic definitions of trigonometric functions in terms of sides of a right triangle.]

1. THEOREM (Pythagorean Identity) For any angle θ, $\sin^2\theta + \cos^2\theta = 1$.

Sketch a right triangle $\triangle ABC$ with right angle at C. Using standard convention, label the sides a, b, and c.

PROOF Let $\triangle ABC$ be the right triangle defined above and let $\theta = \angle A$.

Then $\sin\theta = \underline{\frac{a}{c}}$ and $\cos\theta = \underline{\frac{b}{c}}$ (*)

Also, from the Pythagorean Theorem, we have $\underline{a^2 + b^2 = c^2}$.

Divide both sides by c^2: $\underline{\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1}$

i.e. $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$.

Substitute the trig. functions from (*): $\underline{\sin^2\theta + \cos^2\theta = 1}$. ■

2. THEOREM (Law of Sines). If $\triangle ABC$ is any triangle, then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

[Sketch and label a general triangle.]

PROOF Let $\triangle ABC$ be a triangle. [Show $\frac{\sin A}{a} = \frac{\sin B}{b}$. The proof for the other equality will be similar.]

In any triangle, there can be at most one non-acute angle. So either $\angle A$ or $\angle B$ must be an acute angle.

WLOG, assume that $\angle A$ is acute.

Case 1. $\angle B$ is acute. [Sketch triangle ($\angle A$ and $\angle B$ both acute).]

Drop a perpendicular from C to $\overleftrightarrow{AB}$ and call the foot D. By Lemma 4.8.6, $A * D * B$.

$\sin A = \underline{\dfrac{CD}{b}} \quad \Rightarrow \quad CD = \underline{b \sin A}$ and $\sin B = \underline{\dfrac{CD}{a}} \quad \Rightarrow \quad CD = \underline{a \sin B}$.

Therefore $b \sin A = a \sin B$.

Divide both sides by $ab \Rightarrow \underline{\dfrac{\sin A}{a} = \dfrac{\sin B}{b}}$.

Case 2. $\angle B$ is a right angle. [Finish Case 2 as homework. Note the definitions of $\sin\theta$ and $\cos\theta$ for special angles on p. 116.]

Case 3. $\angle B$ is obtuse. [Sketch triangle ($\angle A$ is acute and $\angle B$ is obtuse).]

Drop a perpendicular from C to $\overleftrightarrow{AB}$ and call the foot D. Since $\angle B$ is obtuse, $A * B * D$.

Note that $\triangle BDC$ is a <u>right triangle</u> with right angle at D.

Also $\angle B = \angle ABC$ is obtuse and forms a <u>linear pair</u> with $\angle BDC$, so they are <u>supplements</u>.

Then $\sin B = \underline{\sin(\angle BDC)}$ by the definition of $\sin\theta$ for obtuse angles on p. 116.

Therefore $\sin B = \underline{\dfrac{CD}{a}} \Rightarrow CD = \underline{a \sin B}$.

Using $\triangle ACD, \sin A = \underline{\dfrac{CD}{b}} \Rightarrow CD = \underline{b \sin A}$.

Therefore $b \sin A = a \sin B$. Divide both sides by $ab \Rightarrow \underline{\dfrac{\sin A}{a} = \dfrac{\sin B}{b}}$.

In all three cases, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$.

Similarly, it can be shown that $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Therefore, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$. ■

Homework: Finish Case 2 for the Law of Sines; Prove the Law of Cosines (Section 5.5, p. 117 #2)