Some theorems we already have:

**Theorem 3.4.7 (Existence and Uniqueness of Angle Bisectors)** If \( A, B, \) and \( C \) are three noncollinear points, then there exists a unique angle bisector for \( \angle BAC \).

**Theorem 3.5.9** If \( l \) is a line and \( P \) is a point on \( l \), then there exists exactly one line \( m \) such that \( P \) lies on \( m \) and \( m \perp l \).

**Theorem 3.5.11 (Existence and Uniqueness of Perpendicular Bisectors)** If \( D \) and \( E \) are two distinct points, then there exists a unique perpendicular bisector for \( DE \).

Some new theorems:

**Theorem 4.1.3 (Existence and Uniqueness of Perpendiculars)** For every line \( l \) and for every point \( P \), there exists a unique line \( m \) such that \( P \) lies on \( m \) and \( m \perp l \).

Sketch picture(s) and explain the difference or similarity between Theorems 3.5.9 and 4.1.3.

**Terminology:** By 4.1.3 we can say, “drop a perpendicular from \( P \) to \( l \).” Also the point \( F \) that where the perpendicular intersects \( l \) is called the foot (of the perpendicular).

**Proof** Let \( l \) be a line and \( P \) be a point.

**Case 1:** \( P \) is on \( l \). Then the conclusion is true by Theorem 3.5.9.

**Case 2:** \( P \) is not on \( l \). [Sketch a picture.] [Existence: Show \( m \) exists s.t. \( P \in m \) and \( m \perp l \).]

Let \( Q \) and \( Q' \) be two distinct points on \( l \) and define the angle \( \angle Q'QP \).

By Angle Construction, there exists a point \( R \) on the opposite side of \( l \) from \( P \) such that \( \angle Q'QP \cong \angle Q'QR \).

By Point Construction, let \( P' \) be a point on \( QR \) such that \( QP \cong QQ' \).

Let \( m = \overrightarrow{PP'} \). [By construction, \( P \in m \), but we still need to show \( m \perp l \).]

By Plane Separation, \( l \cap PP' \neq \emptyset \). Let \( F \) be this point of intersection. Continued →
Subcase (a): $F = Q$. [Resketch (include $R$)]

Then $\angle Q'FP = \angle Q'QP$ and $\angle Q'FP' = \angle Q'QP'$ form a __linear pair__. 

Thus $\mu(\angle Q'QP) + \mu(\angle Q'QP') = 180$.

But since $\angle Q'QP \cong \angle Q'QR$ and $\angle Q'QR = \angle Q'QP'$, then $\angle Q'QP \cong \angle Q'QP'$.

Two congruent angles that sum to 180 must each equal 90. Therefore __m \perp l__.

Subcase (b): $F \neq Q$ and $F$ lies on the ray $\overrightarrow{QQ'}$. [Resketch.]

Then $\angle PQF = \angle PQQ'$ and $\angle P'QF = \angle P'QQ'$.

Thus $\angle PQF \cong \angle P'QF$ since $\angle PQQ' \cong \angle RQQ'$ = $\angle P'QQ'$.

Therefore, $\triangle FQP \cong \triangle FQP'$ by SAS.

Thus $\angle QFP \cong \angle QFP'$ by triangle congruency.

But $\angle QFP$ and $\angle QFP'$ also form a __linear pair__.

Congruent angles that form a linear pair, must be __right angles__ (by the same argument as in subcase (a)).

Therefore $m \perp l$.

Subcase (c): $F \neq Q$ and $F$ lies on the ray opposite $\overrightarrow{QQ'}$. [Resketch.]

Then $\angle PQF$ and $\angle PQQ'$ form a linear pair and are thus __supplements__.

Similarly $\angle P'QF$ and $\angle P'QQ'$ form a linear pair and are supplements.

Therefore, since $\angle Q'QP \cong \angle Q'QP'$, then $\angle PQF \cong \angle P'QF$.

Thus, $\triangle FQP \cong \triangle FQP'$ by __SAS__.

The rest of the proof is the same as subcase (b). Uniqueness: [Homework.]
Theorem 4.3.4 Let $l$ be a line, let $P$ be an external point, and let $F$ be the foot of the perpendicular from $P$ to $l$. If $R$ is any point on $l$ that is different from $F$, then $PR > PF$.

Sketch a picture.

Restate (informal): The _______ distance from a point to a line is measured along the perpendicular.

Proof Homework [Exercise 4.3.7]

[Go on and come back to it, if time.]

Def If $l$ is a line and $P$ is a point, the distance from $P$ to $l$, denoted $d(P,l)$ is defined to be the distance from $P$ to the foot of the perpendicular from $P$ to $l$. 
THEOREM 4.3.6 (POIN TWISE CHARACTERIZATION OF ANGLE BISECTOR) Let \( A, B, \) and \( C \) be three noncollinear points and let \( P \) be a point in the interior of \( \angle BAC \). Then \( P \) lies on the angle bisector of \( \angle BAC \) iff 
\[
d(P, \overrightarrow{AB}) = d(P, \overrightarrow{AC}).
\]
Sketch a picture (one with \( P \) on the angle bisector and one with \( P \) not on it).

PROOF Homework [Exercise 4.3.8] [Go on and come back to it, if time.]

THEOREM 4.3.7 (POINTWISE CHARACTERIZATION OF PERPEND ICULAR BISECTOR) Let \( A \) and \( B \) be distinct points. A point \( P \) lies on the perpendicular bisector of \( \overline{AB} \) iff \( PA = PB \).

Sketch a picture (one with \( P \) on the perpendicular bisector and one with \( P \) not on it).

PROOF Not assigned.

Summary of Homework: Finish Uniqueness part of the proof on p.2; Section 4.3, p. 81: \#(2, 3, 6, 5), 7, 8