Axiom 5 (The Protractor Postulate)

For every angle $\angle BAC$ there is a real number $\mu = \mu(\angle BAC)$ called the measure of $\angle BAC$ such that

1. $0^\circ \leq \mu^\circ < 180^\circ$

2. $\mu = 0^\circ$ iff $\overrightarrow{AB} = \overrightarrow{AC}$

3. For each real number r where $0^\circ < r^\circ < 180^\circ$ and for each of the two half-planes determined by $\overleftrightarrow{AB}$, there exists a unique ray $\overrightarrow{AE}$ such that E is in the half-plane and $\mu(\angle BAE) = r^\circ$

(Angle-Construction Postulate)

4. If the ray $\overrightarrow{AD}$ is between the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\mu(\angle BAC) + \mu(\angle DAC) = \mu(\angle BAC)$

(Angle Addition Postulate)

<u>Def</u> Two angles are **congruent** if they have the ___same angle measure___ .

i.e. $\angle BAC \cong \angle EDF$ if ___$\mu(\angle BAC) = \mu(\angle EDF)$___

<u>Def</u>
An angle with measure $\mu = 90°$ is a ___right___ angle.

An angle with measure $\mu < 90°$ is a ___acute___ angle.

An angle with measure $\mu > 90°$ is a ___obtuse___ angle.

<u>Lemma</u> If A, B, C, and D are four distinct points such that C and D are on the same side of $\overleftrightarrow{AB}$ and D is not on $\overleftrightarrow{AC}$, then either C is in the interior of $\angle BAD$ or D is in the interior of $\angle BAC$.

<u>Proof</u> Let A, B, C and D be defined as above.

Suppose D is not in the interior of $\angle BAC$ [Show ___that C is in the interior of $\angle BAD$.___]

Then D and B are on ___opposite sides___ of $\overleftrightarrow{AC}$

Therefore $\overline{BD} \cap \overleftrightarrow{AC}$ ___$\neq$___ $\emptyset$ by the Plane Separation Postulate.

Let P be this ___unique___ point of intersection.

Then P is between B and D and by theorem 3.3.10 (proved on previous worksheet), ___$\overrightarrow{AB} * \overrightarrow{AP} * \overrightarrow{AD}$___ .

Therefore, P is in the interior of $\angle BAD$. [Still need to show that ___C___ is in the interior of $\angle BAD$.]

[How?: ___Show that $\overrightarrow{AP} = \overrightarrow{AC}$___]

P lies on $\overleftrightarrow{AC}$ since it is the intersection point.

Therefore $\overrightarrow{AP}$ will equal $\overrightarrow{AC}$ as long as they are not ___opposite___ rays.

Since P is in the interior of $\angle BAD$, P and D are on ___the same___ side of $\overleftrightarrow{AB}$.

Then P and C are also on the same side of $\overleftrightarrow{AB}$ and hence, $\overrightarrow{AP}$ and $\overrightarrow{AC}$ ___cannot___ be opposite rays.

Therefore $\overrightarrow{AP} = \overrightarrow{AC}$ $\Rightarrow$ $\overrightarrow{AB} *$ $\overrightarrow{AC}$ $* \overrightarrow{AD}$ $\Rightarrow$ C is in the interior of $\angle BAD$. ■

Why don't we have to prove that "If C is not in the interior of $\angle BAD$, then D is in the interior of BAC?"

<u>THEOREM</u> (Betweenness Theorem for Rays). Let A, B, C, and D be four distinct points such that C and D lie on the same side of $\overleftrightarrow{AB}$. Then $\mu(\angle BAD) < \mu(\angle BAC)$ iff $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

<u>PROOF</u> Let A, B, C, and D be defined as stated above.

$\Leftarrow$: Let $\overrightarrow{AD}$ be between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [Show $\mu(\angle BAD) < \mu(\angle BAC)$]

Then $\underline{\mu(\angle BAD) + \mu(\angle DAC) = \mu(\angle BAC)}$ by the Protractor Postulate Part 4.

Since $\mu(DAC) > 0 \Rightarrow \mu(\angle BAD) < \mu(\angle BAC)$ by the Protractor Postulate <u>Parts 1 & 2</u> .

$\Rightarrow$: Let $\mu(\angle BAD) < \mu(\angle BAC)$ [Show $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.]

BWOC, suppose that $\overrightarrow{AD}$ <u>is not between</u> $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

<u>Case 1</u>: D lies on $\overrightarrow{AC}$.

Then $\mu(\angle BAD)$ $\underline{=}$ $\mu(\angle BAC)$. (*)

<u>Case 2</u>: D does not lie on $\overrightarrow{AC}$.

Then by the previous Lemma, $\underline{C}$ is in the interior of $\underline{\angle BAD}$.

Thus, $\overrightarrow{AC}$ is between $\underline{\overrightarrow{AB} \text{ and } \overrightarrow{AD}}$ by definition of between for rays (Def 3.3.8).

Then by the first half of this theorem, $\underline{\mu(\angle BAC) < \mu(\angle BAD)}$ (**)

Combining the results of Case 1 and 2, we have that $\mu(\angle BAD)\underline{\geq}\mu(\angle BAC)$ $\rightarrow\leftarrow$

Therefore, $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

<u>DEF</u> Let A, B, and C be three noncollinear points. A ray $\overrightarrow{AD}$ is an <u>angle bisector</u> of $\angle BAC$ if D is in the interior of BAC and $\mu(\angle BAD) = \mu(\angle DAC)$.

[Sketch]

<u>THEOREM</u> If A, B, and C are three noncollinear points, then there exists a unique angle bisector for $\angle BAC$.

<u>PROOF</u> Let A, B, and C be three noncollinear points.

By the Protractor Postulate Part 1, $0° \leq \mu(\angle BAC) < 180°$. Thus, $\underline{0°} \leq \frac{1}{2}\mu(\angle BAC) < \underline{90°}$.

Then by the Protractor Postulate Part 3, there exists a unique ray $\overrightarrow{AE}$ such that $\mu(\angle BAE) = \underline{\frac{1}{2}\mu(\angle BAC)}$ and E can be chosen to be on the same side of $\overleftrightarrow{AB}$ as C.

Since $\mu(\angle BAE) = \frac{1}{2}\mu(\angle BAC) < \mu(\angle BAC) \Rightarrow \underline{\overrightarrow{AE}}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ by the Betweenness Theorem for Rays.

[continued on next page]

Then $\mu(\angle BAE) + \mu(\angle EAC) = \mu(\angle BAC)$ by ___the Protractor Postulate Part 4___

$\Rightarrow \frac{1}{2}\mu(\angle BAC) + \mu(\angle EAC) = \mu(\angle BAC) \Rightarrow \mu(\angle EAC) = \frac{1}{2}\mu(\angle BAC) \Rightarrow \mu(\angle BAE) = \mu(\angle EAC) = \frac{1}{2}\mu(\angle BAC)$.

Therefore an $\overrightarrow{AE}$ is an angle bisector for $\angle BAC$.

[Still need to show uniqueness. – do later as homework.]

<u>Def</u> Two lines l and m are ___perpendicular___ if there exists a point A that lies on both l and m and there exist points $B \in l$ and $C \in m$ such that $\angle BAC$ is a right angle.

Denoted: $l \perp m$

[Sketch]

<u>Def</u> A ___perpendicular bisector___ of $\overline{AB}$ is a line l such that the midpoint of $\overline{AB}$ lies on l and $\overleftrightarrow{AB} \perp l$.

[Sketch]

<u>Def</u> Two angles $\angle BAD$ and $\angle DAC$ form a ___linear pair___ if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.

[Sketch]

<u>Def</u> Two angles $\angle BAC$ and $\angle DEF$ are ___supplementary___ if $\mu(\angle BAC) + \mu(\angle DEF) = 180°$.

[Sketch]

<u>Def</u> Angles $\angle BAC$ and $\angle DAE$ form a ___vertical pair___ (or are ___vertical angles___) if $\overrightarrow{AB}$ and $\overrightarrow{AE}$ are opposite rays and $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are opposite rays OR if $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite rays and $\overrightarrow{AC}$ and $\overrightarrow{AE}$ are opposite rays.

[Sketch]

Homework Section 3.4, p. 55: #1[Finish Uniqueness]   Section 3.5, p. 62: #2