Direct Proof

If A, then B

PROOF

1. Start by assuming the hypothesis (If-condition)
   e.g. "Let A" "Suppose A" "Assume A"
   Remember what you want to prove: [Show B]

2. Use Definitions, Axioms, Theorems, Algebra, etc. to write the next line(s).

   Hint: **Definitions almost always give the first line: Ask yourself "What does it mean to be A?"**

   Repeat Step 2 as much as needed:

   - Each new line follows from previous one(s)
   - Keep in mind where you are headed

3. End with Conclusion (Then-condition)
   "Therefore B."

Example

Prove: If $x + 10$ is odd, then x is odd

PROOF

1. Let $x + 10$ be odd. [Show x is odd.]

2. Then by definition, $\exists$ integer k such that $x + 10 = 2k + 1$. [Show $x = 2m + 1$]

   Thus,

$$\begin{aligned} x &= 2k + 1 - 10 \\ &= 2k - 10 + 1 \\ &= 2(k - 5) + 1 \\ &= 2m + 1 \text{ for integer } m = k - 5 \end{aligned}$$

3. Therefore, by definition, x is odd. ■

Proof by Contrapositive (Direct Proof)

If A, then B [Note $A \to B \equiv \sim B \to \sim A$]

PROOF (by contrapositive)

1. Start by assuming not B
   e.g. "Suppose not B"
   [Show not A]

2. Follow steps of Direct Proof to prove not A.

3. Therefore, the contrapositive "If A,then B" is also true.

Example

Prove: If $x + 10$ is odd, then x is odd

PROOF (by contrapositive)

1. Suppose x is not odd. [Show $x + 10$ is not odd.]

2. Then x is even and by definition,
   $\exists$ integer k such that $x = 2k$

$$\begin{aligned} \Rightarrow x + 10 &= 2k + 10 \\ &= 2(k + 5) \\ &= 2m \text{ for integer } m = k + 5 \end{aligned}$$

   Thus, x+10 is even. That is, $x + 10$ is not odd.
   i.e. If x is not odd, then $x + 10$ is not odd

3. Therefore, the contrapositive
   If $x + 10$ is odd, then x is odd
   is also true. ■

Proof by Contradiction (Indirect Proof)

If A, then B

PROOF

1. Let the conditions of A be true.
2. BWOC, suppose not B.
3. Use Definitions, Axioms, Theorems, etc. to create a logical argument until ...

   You reach a contradiction $\longrightarrow\!\!\!\!\!\!\times\;\;$

   [usually contradicts condition A in step 1]

4. "Therefore B"

Example

If $x + 10$ is odd, then x is odd

PROOF

1. Let $x + 10$ be odd.
2. BWOC, suppose x is not odd (i.e. x is even).
3. Then by definition, $\exists$ integer k such that $x = 2k$
   Then

$$\begin{aligned} x + 10 &= 2k + 10 \\ &= 2(k+5) \\ &= 2m \text{ for integer } m = k+5 \end{aligned}$$

   Thus, $x + 10$ is even $\longrightarrow\!\!\!\!\!\!\times\;\;$

4. Therefore, x must be odd. $\blacksquare$

Proof by Induction

Prove that a proposition P_n is true for all n
(or all $n \geq m$)

PROOF

Basis ($n = 1$ or $n = m$): Verify that the proposition holds for $n = 1$ or m.

Induction: Assume true for $n = k$. i.e. P_k is true.

[Show that it holds for $n = k + 1$

i.e. Show that P_{k+1} is true.]

- Start with one side of P_{k+1} statement
- Use algebra to manipulate so that you can....
- ... ***Use the Induction Assumption!!***
- More manipulation, if needed to look like the other side of P_{k+1} statement.

Thus P_{k+1} is true.
Therefore, by Mathematical Induction,
the proposition P_n is true for all n (or $n \geq m$).

Example

Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2 \qquad \forall n \in \mathbb{N}$

PROOF

Basis ($n = 1$): $1 = 1^2$ $\checkmark$

Induction: Assume true for $n = k$.
i.e. $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ is true.

[Show that $1 + 3 + \cdots + (2k-1) + [2(k+1) - 1] = (k+1)^2$]

$$\begin{aligned} 1 + 3 + \cdots + (2k-1) + [2(k+1) - 1] & \\ &= k^2 + [2(k+1) - 1] \\ &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

i.e. $1 + 3 + \cdots + (2k-1) + [2(k+1) - 1] = (k+1)^2$

Thus it holds for $n = k + 1$

Therefore, by Mathematical Induction,
$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \qquad \forall n \in \mathbb{N}$.

Disproof by Counterexample

Show that "If A, then B" is false by counterexample, by finding specific values that hold for A, but are false for B.

Hint: Counterexample values are often found at strange or interesting points, e.g. zero, negative numbers, end points, etc.

Example: If $a < b$, then $ac < bc$. Let $a = 2, b = 3$, and $c = -4$ so that $a < b$. But $ac = 2(-4) = -8 \not< -12 = 3(-4) = bc$.