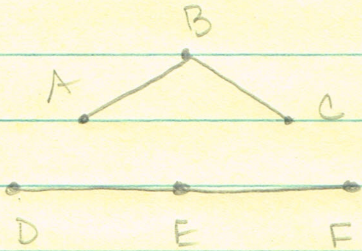
Exam1 Key.

1. See Exam

2. Points: $\{A, B, C, D, E, F\}$
   Lines: $\{A, B\}$, $\{B, C\}$, $\{D, E, F\}$

(a)



(b)

IA1	Fails	eg Pts A + C do not lie on a line
IA 2	Holds	
IA 3	Holds	
Euc PP	Fails	eg line $\overleftrightarrow{AB}$ and pt $C \notin \overleftrightarrow{AB}$
Ell PP	Fails	eg line $\overleftrightarrow{AB}$ + pt $D \notin \overleftrightarrow{AB}$
Hyp PP	Fails	eg line $\overleftrightarrow{AB}$ + pt $D \notin \overleftrightarrow{AB}$

3. (a) Contrapositive: If $l \cap m$ does not contain only one point, then l and m are parallel or the same line.

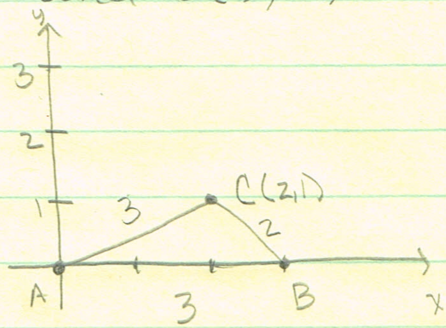
(b) Negation: Two angles form a linear pair, and (but) the sum of their angle measures is not 180.

(c) Negation: There exists a Monday holiday and there is school.

4. P iff Q $(P \Leftrightarrow Q)$ is $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$

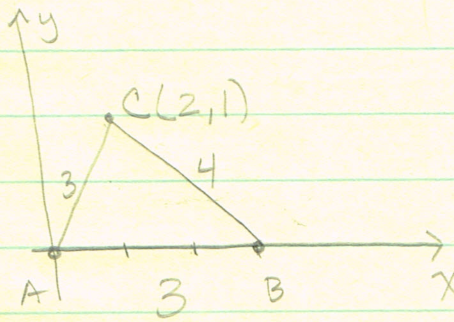
P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

5. (a) $A(0,0)$ and $B(3,0)$ [Two common answers]



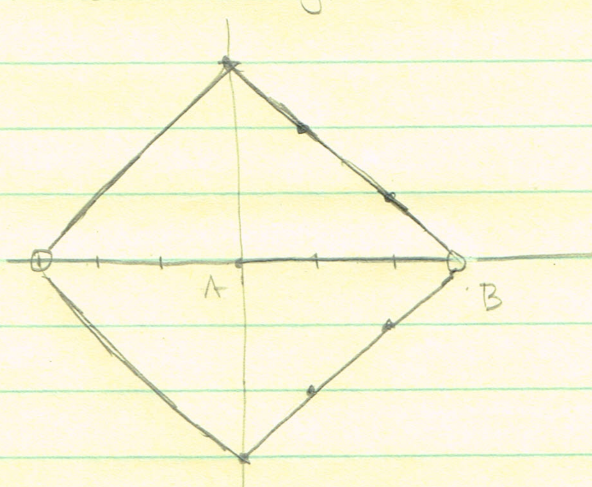
$\rho(BC) = 2$

OR



$\rho(B,C) = 4$

(b) No, it is not unique. All pts C that would create an isosceles triangle are 3 units away from $A(0,0)$.



ie circle in the taxicab metric w/ radius 3. But the points $(-3,0)$ and $(3,0)$ are excluded since they do not yield a triangle

6. (a) If l is any line, then there exists lines m and n s.t. $l, m, \& n$ are distinct and both $m \& n$ intersect l.

Proof. Let l be a line.

By IA2, there exist two points $P \& Q$ on l. i.e. $l = \overleftrightarrow{PQ}$

By Theorem 2.6.3, there exists a point R s.t. R is not on l.

Let $m = \overleftrightarrow{PR}$ and $n = \overleftrightarrow{QR}$.

Since P is on $l \& m$, l intersects m.

Since Q is on $l \& n$, l intersects n. [Show $l, m, \& n$ are distinct]

Since R does not lie on l, but is on $m \& n$ $\Rightarrow l \neq m$, $l \neq n$.

[Show $m \neq n$]

BWOC, Suppose $m = n$. Then $P, Q, \& R$ all lie on $m = n$.

In particular $P \& Q$ are a distinct pair of points on $m = n$

$\Rightarrow$ By IA1, $m = l$ $\times$

$\therefore m \neq n$. ∎

6.(b) Prove equality of sets: $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.

Proof. [Show $\overrightarrow{AB} \cap \overrightarrow{BA} \subseteq \overline{AB}$ and $\overline{AB} \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}$]

Let $P \in \overrightarrow{AB} \cap \overrightarrow{BA}$ [Show $P \in \overline{AB}$]

Then $P \in \overrightarrow{AB}$ and $P \in \overrightarrow{BA}$.

$\Rightarrow$ ($P = A$ or $P = B$ or $A * P * B$ or $A * B * P$) AND ($P = A$ or $P = B$ or $B * P * A$ or $B * A * P$)

$\Rightarrow$ $P = A$ or $P = B$ or $A * P * B$ ← same

In all three cases $P \in \overline{AB}$ by definition.

$\therefore$ $\overrightarrow{AB} \cap \overrightarrow{BA} \subseteq \overline{AB}$ (*)

Let $P \in \overline{AB}$. [Show $P \in \overrightarrow{AB} \cap \overrightarrow{BA}$]

Then $P = A$ or $P = B$ or $A * P * B$.

$\Rightarrow$ $P \in \overrightarrow{AB}$ and $P \in \overrightarrow{BA}$ by definition of ray.

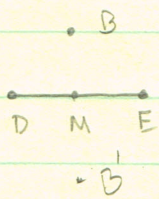
$\Rightarrow$ $P \in \overrightarrow{AB} \cap \overrightarrow{BA}$

$\therefore$ $\overline{AB} \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}$ (**)

By (*) + (**) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$ ∎

6(c). If D and E are two distinct points, then there exists a unique perpendicular bisector for $\overline{DE}$.

Show unique $\perp$ bisector for $\overline{DE}$
ie Show $\exists$ line l s.t.
① $l \perp \overleftrightarrow{DE}$ $(90°)$
② l bisects $\overline{DE}$
ie l goes through midpt.



Proof Let D & E be two distinct points. By Theorem 3.2.22 there exists a unique midpoint M of DE.

$\Rightarrow$ $DM = ME$. $(*)$

Since $90° \in (0°, 180°)$, there exists a point B not on $\overleftrightarrow{DE}$ st. $\mu(\angle EMB) = 90°$ $(**)$ by the Angle-Construction Postulate.

Let $l = \overleftrightarrow{MB}$.

Then by $(*)$ & $(**)$ l is a perpendicular bisector of $\overline{DE}$ (by definition).

Uniqueness: The midpoint M is unique. But the ray $\overrightarrow{MB}$ is not unique since a 2^{nd} ray in the other half-plane exists, call in $\overrightarrow{MB'}$. But these 2 rays are unique in each half-plane and they are opposite rays $\therefore$ $l = \overleftrightarrow{MB} = \overleftrightarrow{MB'}$ is unique.

Traditional Uniqueness: Suppose there is another perp. bisector n to $\overline{DE}$. Then $M \in n$. Also by the Angle-Construction Postulate, there is a pt. C on the same side a B, s.t. $\mu(\angle EMC) = 90°$ and the ray $\overrightarrow{MC}$ is unique. $\therefore$ $\overrightarrow{MC} = \overrightarrow{MB}$

$\Rightarrow$ $n = l$ ∎

2. If A and B are distinct points, then there exists a unique point M such that M is the midpoint of $\overline{AB}$.

Proof Let A and B be two distinct points.
Let $l = \overleftrightarrow{AB}$.
Let a & b be the coordinates of A & B, respectively.
Define $m = \frac{a+b}{2}$ which is the coordinate of a point M on l.
Since m is the midpt of 2 real numbers. $\Rightarrow$
$a < m < b$ or $b < m < a$ by properties of real numbers.
Therefore $A * M * B$ by Betweenness for Points.
Also
$AM = |a-m| = |a - \frac{a+b}{2}| = |\frac{2a-a-b}{2}| = |\frac{a-b}{2}|$ and
$MB = |m-b| = |\frac{a+b}{2} - b| = |\frac{a+b-2b}{2}| = |\frac{a-b}{2}|$
$\therefore$ By definition M is a midpoint of $\overline{AB}$.

Uniqueness: BWOC, suppose N is also a midpt of $\overline{AB}$ with $N \neq M$. Let n be the coordinate of N.
Then $|a-n| = |n-b| = |\frac{a-b}{2}|$ and $a < n < b$ and $a < m < b$ or vice versa.
$\Rightarrow |a-n| = |a-m|$
$\Rightarrow n = m$
$\Rightarrow N = M$ $*$
$\therefore$ The midpoint is unique.

Exam 1 Take-Home

1. Let A + B be 2 distinct points. Prove that the set $\overline{AB}$ is a convex set.

<u>Proof.</u> Let A and B be 2 distinct points. [Show $\overline{AB}$ is convex]
Let a and b be the coordinates of A and B.
Then $\overline{AB} = \{ P \in \overleftrightarrow{AB} \mid a \leq p \leq b$ where p is the coord of $P \}$.
Let $C, D \in \overline{AB}$ [$\underbrace{\text{Shows } \overline{CD} \text{ in } \overline{AB}}_{\text{from def of convex}}$ ie $\overline{CD} \subseteq \overline{AB}$]
Let c + d be the coordinates.
WLOG, assume $c < d$.
Then $a \leq c < d \leq b$ (*) by definition of $\overline{AB}$

Let $P \in \overline{CD}$ [Shows $P \in \overline{AB}$ to show subset $\overline{CD} \subseteq AB$]
Then $c \leq p \leq d$ (**) by def. of $\overline{CD}$

Combining (*) and (**) $\Rightarrow a \leq c \leq p \leq d \leq b$
$\Rightarrow a \leq p \leq b$
$\Rightarrow P \in \overline{AB}$ by definition of $\overline{AB}$
$\therefore \overline{CD} \subseteq \overline{AB}$
<u>ie</u> $\overline{CD}$ is in $\overline{AB}$ and therefore AB is convex.