Exam 1 Key

1. $a_{n+1} = -a_n - 4$, $a_0 = 3$

(a) $a = \frac{b}{1-r} = \frac{-4}{1-(-1)} = \frac{-4}{2} = \boxed{-2}$

OR $a = -a - 4 \Rightarrow 2a = -4 \Rightarrow a = -2$ ✓

(b) $r = -1 \Rightarrow$ $\boxed{\text{Neutrally Stable}}$

(c) $a_n = Cr^n + \frac{b}{1-r}$

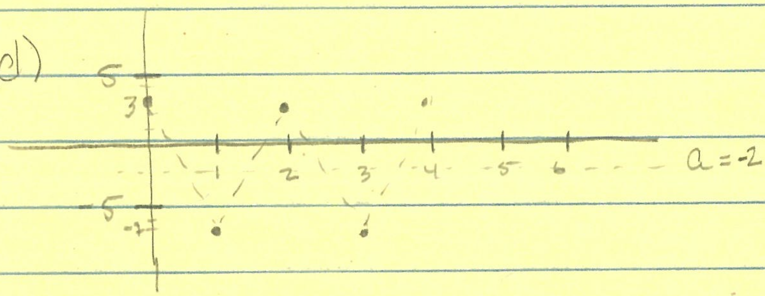
$a_n = C(-1)^n - 2$

$a_0 = C(-1)^0 - 2 = 3$ $\rightarrow$ $C - 2 = 3$

$C = 5$

$\boxed{a_n = 5(-1)^n - 2}$

(d)



$\boxed{\text{Steady oscillations about } a = -2}$

2. (a) $\Delta a_n = 0.008 a_n - 1200$

$a_{n+1} - a_n = 0.008 a_n - 1200$

$$\boxed{\begin{array}{l} a_{n+1} = 1.008 a_n - 1200 \\ a_{240} = 0 \end{array}}$$

(b) $a = \frac{b}{1-r} = \frac{-1200}{1-1.008} = 150{,}000$

$a_n = C(1.008)^n + 150000$

$a_{240} = C(1.008)^{240} + 150000 = 0$

$\Rightarrow C = \frac{-150000}{(1.008)^{240}} \approx -22159.68$

$$\boxed{\begin{array}{l} a_n = \frac{-150000}{(1.008)^{240}}(1.008)^n + 150000 \\ \text{OR} \quad a_n \approx -22159.68(1.008)^n + 150000 \end{array}}$$

(c) $a_0 = -22159.68(1.008)^0 + 150000$

$= \boxed{\$127{,}840.32}$

3. $\Delta S_n = \qquad -0.001\, I_n S_n$

$\Delta I_n = -0.6\, I_n + 0.001\, I_n S_n$

$\Delta R_n = 0.6\, I_n$

(a) • For Susceptibles S_n, the $-0.001\, I_n S_n$ term indicates that when Susceptibles and Infected persons interact some of the Susceptible population decreases as they become part of the Infected population.

• For the Infected I_n, the $+0.001$ term gives the increase in Infected due to interactions with Susceptibles. The $-0.6\, I_n$ term indicates that 60% of the Infected population are removed, while 40% remain infected.

• For Removed R_n, the $+0.6\, I_n$ term is the 60% of Infected population who are now immune because the disease has run its course and they have recovered or died.

• The 0.6 & 0.001 coefficients show up in two equations because the overall pop. size doesn't change and leaving one population means entering another in the same proportion

(b) (1) $0 = -.001\, I \cdot S$

(2) $0 = -0.6\, I + 0.001\, I \cdot S$

(3) $0 = 0.6\, I$

$\Rightarrow$ From (3): $I = 0$, then note (1) & (2) are automatically satisfied for any R and S.

$\boxed{\text{So Equil is } (S, 0, R) \text{ with } S+0+R = \boxed{S+R = 1000}}$

4. $\lambda = 1.1,\ 0.8 \qquad A = \begin{bmatrix} 0.5 & 0.6 \\ -0.3 & 1.4 \end{bmatrix}$

(a) $\lambda_1 = 1.1 > 1 \qquad 0 < \lambda_2 = 0.8 < 1$

So $(0,0)$ is a $\boxed{\text{saddle point}}$

(b) $\underline{\underline{\lambda_1 = 1.1}} \Rightarrow \begin{bmatrix} 0.5-1.1 & 0.6 \\ -0.3 & 1.4-1.1 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.6 \\ -0.3 & 0.3 \end{bmatrix} \Big\} \Rightarrow$ same eqn

$\Rightarrow -0.6x + 0.6y = 0 \Rightarrow -x + y = 0 \Rightarrow y = x \Rightarrow$ $\boxed{\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ direction of repulsion } (\lambda_1 = 1.1)}$

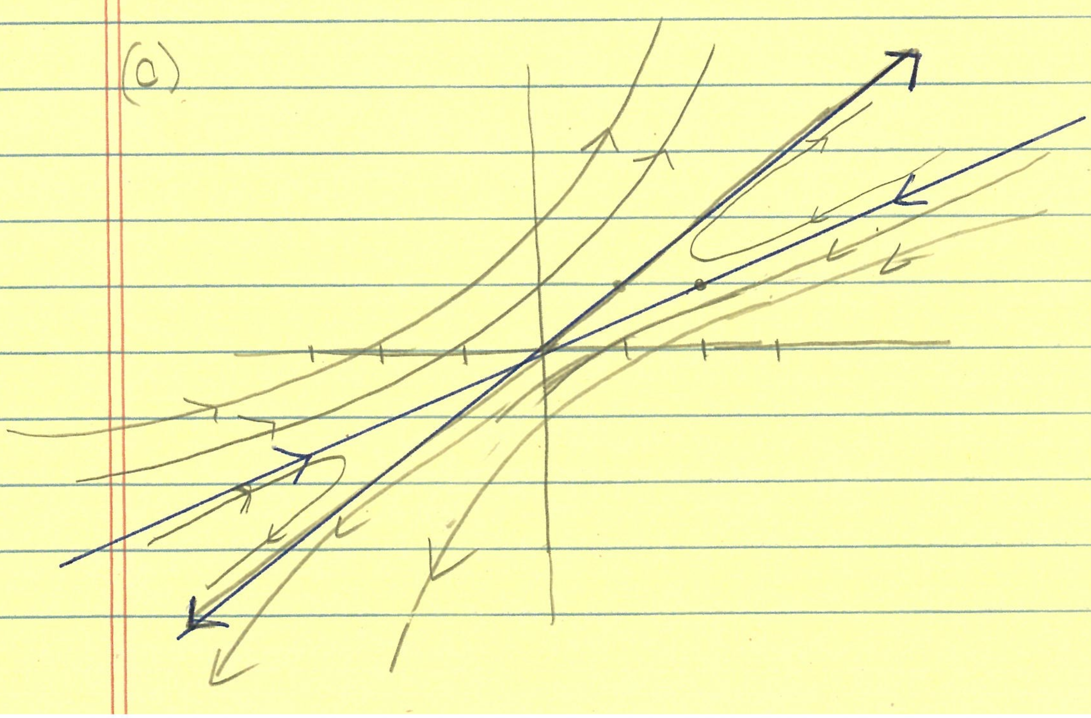
$\lambda = 0.8 \Rightarrow \begin{bmatrix} 0.5-0.8 & 0.6 \\ -0.3 & 1.4-0.8 \end{bmatrix}$

$= \begin{bmatrix} -0.3 & 0.6 \\ -0.3 & 0.6 \end{bmatrix} \Big\} \Rightarrow$ same eqn

$-0.3x + 0.6y = 0 \Rightarrow x - 2y = 0 \Rightarrow x = 2y \Rightarrow$

$\boxed{V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is direction of attraction } (\lambda = 0.8)}$

(c)



5. $j_{n+1} = 1.6 a_n$

$a_{n+1} = 0.3 j_n + 0.8 a_n$

(a) $\boxed{A = \begin{bmatrix} 0 & 1.6 \\ 0.3 & 0.8 \end{bmatrix}}$ for $\vec{x}_{n+1} = A\vec{x}_n$ and $\vec{x}_n = \begin{bmatrix} j_n \\ a_n \end{bmatrix}$

(b) $\vec{X}_n = c(\lambda_1)^n \vec{V}_1 + c(\lambda_2)^n \vec{V}_2$ ⇐ Need λ's & $\vec{V}$'s

$$\begin{vmatrix} 0-\lambda & 1.6 \\ 0.3 & 0.8-\lambda \end{vmatrix} = (-\lambda)(0.8-\lambda) - 1.6(0.3) = 0$$

$$\lambda^2 - 0.8\lambda - .48 = 0$$

$$\lambda^2 = \frac{0.8 \pm \sqrt{(.8)^2 - 4(1)(-.48)}}{2(1)}$$

$$= \frac{0.8 \pm 1.6}{2} = 1.2 \text{ or } -0.4$$

$\underline{\lambda_1 = 1.2}$ $\begin{bmatrix} -1.2 & 1.6 \\ 0.3 & 0.8-1.2 \end{bmatrix}$

$= \begin{bmatrix} -1.2 & 1.6 \\ 0.3 & -0.4 \end{bmatrix} \Rightarrow$ same eqn.

$-1.2x + 1.6y = 0 \Rightarrow 1.6y = 1.2x \Rightarrow y = \frac{1.2}{1.6}x \Rightarrow y = \frac{3}{4}x \Rightarrow \vec{V}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\lambda_2 = -0.4$ $\begin{bmatrix} +0.4 & 1.6 \\ 0.3 & 0.8+0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.4 & 1.6 \\ 0.3 & 1.2 \end{bmatrix} \Rightarrow$ Same eqn $\Rightarrow 0.4x + 1.6y = 0$

$\Rightarrow 1.6y = -0.4x \Rightarrow y = \frac{-0.4x}{1.6} \Rightarrow y = -\frac{1}{4}x \Rightarrow \vec{V}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$$\boxed{\vec{X}_n = C_1(1.2)^n \begin{bmatrix} 4 \\ 3 \end{bmatrix} + C_2(-0.4)^n \begin{bmatrix} 4 \\ -1 \end{bmatrix}}$$

(c) $\vec{X}_0 = C_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix} \Rightarrow \begin{matrix} 4C_1 + 4C_2 = 8 \\ 3C_1 - C_2 = 10 \end{matrix} \Rightarrow \begin{matrix} 4C_1 + 4C_2 = 8 \\ 12C_1 - 4C_2 = 40 \end{matrix}$

$\Rightarrow \boxed{\vec{X}_n = 3(1.2)^n \begin{bmatrix} 4 \\ 3 \end{bmatrix} - (-0.4)^n \begin{bmatrix} 4 \\ -1 \end{bmatrix}}$

$16C_1 = 48$

$C_1 = 3$

$3(3) - C_2 = 10$

$-C_2 = 1$

$C_2 = -1$

(d) For large $n \to (-0.4)^n \to 0$

So $\vec{x}_n \to 3(1.2)^n \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

The populations both grow by 20% each year and approach a ratio of 4 juveniles to 3 adults.

6. (a) See Excel

(b) Yes, the system is sensitive to IC's.
Although the number of infected always goes to the equilibrium value $I = 0$, the equilibrium values for S & R change based on the IC's.

(c) • The longterm behavior always results in $I \to 0$ and $S, R \to$ positive equilibrium values, so the equilibrium is stable.
• S_n & R_n appear to change similar to a logistic curve, but I_n goes up and then decreases to 0.
• No, the entire population never becomes infected even if you increase the initial number of infected.

(d) See Excel.