Name: ________________________________ **Exam 1**

Math 311 Mathematical Modeling – Crawford **16 March 2017**

Score	
1	/16
2	/16
3	/12
4	/16
5	/22
6	/22
Total	/100

Part I:

- Books, notes (in any form), and computers are not allowed, but you may use a calculator.
- Clearly indicate your answers.
- You must turn in Part I before beginning on Part II.
- ***Show all your work*** – partial credit may be given for written work.

ALL WORK AND ANSWERS SHOULD BE ON SEPARATE PAPER.
USE THIS SHEET AS A COVER SHEET.

Good Luck!

1. (16 pts). Given the dynamical system $a_{n+1} = -a_n - 4,\ a_0 = 3,$

(a). Find the equilibrium value, if it exists.

(b). Classify the equilibrium as stable, unstable, or neutrally stable.

(c). Find the analytical solution.

(d). Describe the long-term behavior for the given initial condition a_0. [e.g. Is it growing/decreasing? Does it oscillate (about what)? Is it moving towards or away from a limiting value? etc.]

2. (16 pts). Your grandparents have an annuity. The value of the annuity increases each month by 0.8% interest as the previous month's balance is deposited. Your grandparents withdraw $1200 each month for living expenses. They would like the annuity to last 20 years.

(a). Formulate a DDS model that represents the annuity being depleted in 20 years (240 months).

(b). Find the *analytical* solution to the DDS.

(c). Use the analytical solution to determine how much they must have in the annuity initially for it to be depleted in 20 years.

3. (12 pts). The following is an SIR model for the spread of a disease. At time n, S_n is the number of people susceptible to the disease and I_n is the number of people infected and are able to spread the disease. R_n is the number of people who have been removed because they are immune to the disease due to recovery, vaccination, or death. The total population $S_n + I_n + R_n$ remains fixed at 1000 for all n.

$$\begin{aligned} \Delta S_n &= & &- 0.001\, I_n S_n \\ \Delta I_n &= -0.6\, I_n & &+ 0.001\, I_n S_n \\ \Delta R_n &= 0.6\, I_n & & \end{aligned}$$

(a). Clearly explain what the coefficients and their signs represent in terms of the three populations and why it makes sense. For example, explain what happens with and without interactions between the populations. Explain why it makes sense for some of the coefficients to be the negative of each other.

(b). Solve for the equilibrium values (S, I, R). If there is not enough information to fully determine the equilibrium, clearly state any conditions that must hold.

4. (16 pts). Given that $\lambda = 1.1, 0.8$ are the eigenvalues of $A = \begin{bmatrix} .5 & .6 \\ -.3 & 1.4 \end{bmatrix}$,

(a). Classify the origin as an attractor, repeller, or saddle point of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

(b). Find the directions of greatest attraction and/or repulsion.

(c). Make a sketch showing the directions of greatest attraction or repulsion. Include several typical trajectories with arrows to indicate the direction of flow.

5. (22 pts). A stage-matrix model for an animal species describes how the females in a population transition to different life stages. Let j_n represent the number of juveniles and a_n represent the number of adults at year n. Consider the following stage model.

$$\begin{aligned} j_{n+1} &= & & 1.6 a_n \\ a_{n+1} &= 0.3 j_n & &+ 0.8 a_n \end{aligned}$$

(a). Construct the stage-matrix A such that $\mathbf{x}_{n+1} = A\mathbf{x}_n$, for $n \geq 0$.

(b). Find the general solution of the system analytically.

(c). If $j_0 = 8, a_0 = 10$, find the solution to the system.

(d). Describe the long-term (sufficiently large n) behavior of the solution. For example, determine whether the population stages are growing or decreasing. If so, by how much? If they eventually approach a steady ratio of juveniles to adults, what is the ratio?

Part II: You must turn in Part I before beginning Part II of the exam. Books and notes (in any form) are not allowed. You may use a computer and calculator for Part II. Clearly indicate your answers.

6. (22 pts). The following is an SIR model for the spread of a disease. At time n, S_n is the number of people susceptible to the disease and I_n is the number of people infected and are able to spread the disease. R_n is the number of people who have been removed because they are immune to the disease due to recovery, vaccination, or death. The total population $S_n + I_n + R_n$ remains fixed at 1000 for all n.

$$\begin{aligned}
\Delta S_n &= & &- 0.001\, I_n S_n \\
\Delta I_n &= -0.6\, I_n & &+ 0.001\, I_n S_n \\
\Delta R_n &= 0.6\, I_n & &
\end{aligned}$$

(a). Use Excel to iterate the system for each of the following initial conditions and graph the solutions. The total population, including Removed, should always be 1000.

	Susceptible	Infected	Removed
Case 1	999	1	0
Case 2	990	10	0
Case 3	900	100	0

(b). Is the system sensitive to initial conditions? ***Briefly*** explain.

(c). Describe the qualitative behavior of the solutions. For example, does there appear to be an equilibrium solution? If so, is it stable or unstable? Do the solutions oscillate? If so, about what? Will the entire population eventually become infected? If not, will increasing the initial number infected result in the entire population eventually becoming infected?

(d). Suppose part of the initial population receives a vaccination against the disease. Iterate the system for the following initial conditions and graph the solutions. Compare them with Case 2 above and explain what effect immunization has on the spread and recovery of the disease.
[It may be helpful to graph just the Infected Population for Case 2 and 2b together.]

	Susceptible	Infected	Removed
Case 2b	890	10	100

[Make sure any graphs have a legend. Save your Excel file as Exam1_LastNameFirstNameOptional.xlsx. Email it to crawford@elmhurst.edu.]