Name: Key

Math 301 Discrete Mathematics – Crawford

Quiz 3

17 April 2019

Books, calculators, and notes (in any form) are not are allowed. Show all your work for credit. ***Good luck!***

1. (3 pts) Find the first four terms of the following recursively defined sequence.

$b_k = 2b_{k-1} + 3b_{k-2}$ for every integer $k \geq 2$
$b_0 = -2$
$b_1 = 1$

$b_2 = 2b_1 + 3b_0 = 2(1) + 3(-2) = 2 - 6 = -4$

$b_3 = 2b_2 + 3b_1 = 2(-4) + 3(1) = -8 + 3 = -5$

$$\boxed{\begin{aligned} b_0 &= -2 \\ b_1 &= 1 \\ b_2 &= -4 \\ b_3 &= -5 \end{aligned}}$$

2. (6 pts) Given the sequence defined recursively below, use iteration to guess an explicit formula for the sequence. If possible, use any of the formulas given at the bottom of the page to simplify your answer.

$d_k = 4d_{k-1} + 1$, for each integer $k \geq 2$
$d_1 = 1$.

$d_2 = 4d_0 + 1 = 4(1) + 1 = 4 + 1$

$d_3 = 4(4+1) + 1 = 4^2 + 4 + 1$

$d_4 = 4(4^2 + 4 + 1) + 1 = 4^3 + 4^2 + 4 + 1$

$\vdots$

$$d_n = \underbrace{4^{n-1} + 4^{n-2} + \cdots + 4 + 1}$$

$$= \sum_{i=0}^{n-1} 4^i = \frac{4^n - 1}{4 - 1} = \boxed{\frac{4^n - 1}{3}}$$

Formulas that may or may not be helpful: $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ $\quad \displaystyle\sum_{i=0}^{n} r^i = 1 + r + r^2 + r^3 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$

3. (6 pts) Suppose that $a_0, a_1, a_2, \ldots$ is a sequence defined as follows:

$a_0 = 3, a_1 = 10,$
$a_k = 7a_{k-1} - 12a_{k-2}$ for all integers $k \geq 2$. (*)

Use strong induction to prove that $a_n = 4^n + 2 \cdot 3^n$ for all integers $n \geq 0$.

Proof. Let $P(n)$ be the formula $a_n = 4^n + 2 \cdot 3^n$ $\forall$ integers $n \geq 0$.

Basis Step

$P(0)$: $a_0 = 3$ (given) and $a_0 = 4^0 + 2 \cdot 3^0 = 1 + 2 = 3$ ✓

$P(1)$: $a_1 = 10$ (given) and $a_1 = 4^1 + 2 \cdot 3^1 = 4 + 6 = 10$ ✓

Induction Step Assume $P(i)$ is true for $i = 0, 1, 3, \ldots, k$ for $k \geq 1$

ie $a_i = 4^i + 2 \cdot 3^i$ for $i = 0, 1, 2, \ldots k$ and $k \geq 1$.

[Show $P(k+1)$ is true. ie Show $a_{k+1} = 4^{k+1} + 2 \cdot 3^{k+1}$]

Since $k \geq 1 \Rightarrow k+1 \geq 2$ and the definition formula (*) applies.

$$a_{k+1} = 7a_k - 12_{k-1}$$
$$= 7(4^k + 2 \cdot 3^k) - 12(4^{k-1} + 2 \cdot 3^{k-1})$$ by the Induction Hypothesis.
$$= 7 \cdot 4^k + 7 \cdot (2 \cdot 3^k) - 12 \cdot (4^{k-1}) - 12(2 \cdot 3^{k-1})$$
$$= 7 \cdot 4^k + 7 \cdot (2 \cdot 3^k) - 3 \cdot 4(4^{k-1}) - 4 \cdot 3(2 \cdot 3^{k-1})$$
$$= 7 \cdot 4^k + 7 \cdot (2 \cdot 3^k) - 3 \cdot 4^k - 4 \cdot (2 \cdot 3^k)$$
$$= 4 \cdot 4^k + 3 \cdot (2 \cdot 3^k)$$
$$a_{k+1} = 4^{k+1} + 2 \cdot 3^{k+1}$$

ie $P(k+1)$ is true

$\therefore$ By strong mathematical induction, $a_n = 4^n + 2 \cdot 3^n$ for $n \geq 0$ ∎