Name: ______________________________ **Exam 2**

Math 301 Discrete Mathematics – Crawford **1 May 2019**

Books, notes (in any form), and calculators are not allowed. ***Put all of your work and answers on other sheets of paper.*** Include this sheet as a cover sheet. ***Show all your work.*** Partial credit may be given for written work. Good Luck!

1. (12 pts) Given the following algorithm, make a trace table and clearly state the final values of $j, s,$ and t.

$j := 3$
$s := 18$
$t := 4$
while $j \neq 7$
  if $(j > 5$ or $j = 3)$
    then $s := s - 3$
    else $t := 2t + j$
  $j := j + 1$
end while

	0	1	2	3	4
j	3	4	5	6	7
s	18	15	(15)	(15)	12
t	4	(4)	12	29	(29)

$\boxed{j = 7 \quad s = 12 \quad t = 29}$

2. (12 pts) Let $a_0 = -1, a_1 = 2, a_2 = -2, a_3 = 3, a_4 = -2, a_5 = 2,$ and $a_6 = -1$. Compute each of the following:

(a). $\sum_{k=0}^{6} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6$

$= -1 + 2 - 2 + 3 - 2 + 2 - 1 = \boxed{1}$

(b). $\sum_{j=1}^{3} a_{2j} = a_2 + a_4 + a_6 = -2 - 2 - 1 = \boxed{-5}$

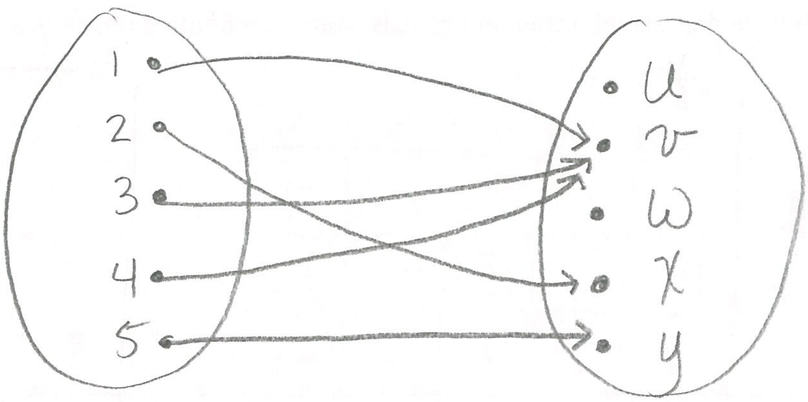
(c). $\prod_{i=0}^{3} a_i = a_0 \cdot a_1 \cdot a_2 \cdot a_3 = (-1)(2)(-2)(3) = \boxed{12}$

(d). $\prod_{k=1}^{3} k^2 = (1)^2(2)^2(3)^2 = \boxed{36}$

3. (12 pts) Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{u, v, w, x, y\}$ and define $h : X \to Y$ as follows:

$$h(1) = v,\ h(2) = x,\ h(3) = v,\ h(4) = v,\ h(5) = y.$$

(a). Draw an arrow diagram for h.



(b). Let $A = \{1, 2\}, C = \{x, v\}, D = \{w\}$. Find $h(A), h^{-1}(C), h^{-1}(D)$.

$h(A) = \boxed{\{v, x\}}$

$h^{-1}(C) = \boxed{\{1, 2, 3, 4\}}$

$h^{-1}(D) = \boxed{\emptyset}$

4. (10 pts) Define a relation P on $\mathbb{Z}$ as follows: For every ordered pair $(m, n) \in \mathbb{Z} \times \mathbb{Z}$,

$m\ P\ n$ iff m and n have a common prime factor. [Justify your answers.]

(a). Is $15\ P\ 25$?

Yes, since 5 is a factor of both 15 & 25.

(b). Is $0\ P\ 5$?

Yes, since 5 is a factor of both 0 and 5.

(c). Is $22\ P\ 27$?

No since the only factors of 22 are 1, 2, 11, 22 and the only factors of 27 are 1, 3, 9, 27. The only shared factor is 1 which is not prime (by def.)

5. (14 pts) Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 2n + 5$, for each integer n.

(a). Is g one-to-one? Prove or give a counterexample.

Yes.

Proof. Let g be defined as above and let $n_1, n_2 \in \mathbb{Z}$ s.t. $g(n_1) = g(n_2)$. [Show $n_1 = n_2$]

Then $2n_1 + 5 = 2n_2 + 5$

$\Rightarrow 2n_1 = 2n_2$

$\Rightarrow n_1 = n_2$

$\therefore$ g is one-to one (by def.) $\blacksquare$

(b). Is g onto? Prove or give a counterexample.

No. Let $0 \in \mathbb{Z}$ (co-domain) ← There are many choices for a counterexample

Then if g is onto $\mathbb{Z}$, $\exists n \in \mathbb{Z}$ s.t. $g(n) = 0$

ie $2n + 5 = 0$

$\Rightarrow 2n = -5$

$n = -\frac{5}{2}$ which is not an integer

$\therefore$ Not onto.

6. (14 pts) Prove by contradiction: For any even integer n, $n^2 - 2$ is not divisible by 4.

Proof. Let n be an even integer.

BWOC, suppose $4 \mid (n^2 - 2)$.

Then $n^2 - 2 = 4r$ for some integer r.

Also since n is even $n = 2k$ for some integer k.

$\Rightarrow (2k)^2 - 2 = 4r$

$\Rightarrow 4k^2 - 2 = 4r$

$\Rightarrow 4k^2 - 4r = 2$

$\Rightarrow 4(k^2 - r) = 2$

$\rightarrow k^2 - r = \frac{1}{2}$ $\times$ since $k^2 - r$ is an integer by closure properties

$\therefore 4 \nmid (n^2 - 2)$ $\blacksquare$

7. (14 pts) Prove by Mathematical Induction:

For every integer $n \geq 1$, $\quad 1 + 6 + 11 + 16 + \cdots (5n-4) = \dfrac{n(5n-3)}{2}$

Proof. Let $P(n)$ be the formula $1+6+11+16+\ldots+(5n-4) = \dfrac{n(5n-3)}{2}$

Basis Step ($n=1$): LHS $= 1$ $\quad$ RHS $= \dfrac{1(5-3)}{2} = \dfrac{2}{2} = 1$ $\quad$ ie $P(1)$ is true.

Induction step: Assume $P(k)$ is true ie $1+6+11+\ldots+(5k-4) = \dfrac{k(5k-3)}{2}$

$$(k+1)(5k+2) = \frac{5k^2+7k+2}{2} \quad (*)$$

[Show $P(k+1)$ is true. ie Show $1+6+11+\ldots+(5k-4)+(5(k+1)-4) = \dfrac{(k+1)(5(k+1)-3)}{2}$]

LHS of $P(k+1)$: $1+6+11+\ldots+(5k-4)+(5(k+1)-4) = \dfrac{k(5k-3)}{2} + (5(k+1)-4)$ by the Induction Assumption

$$= \frac{k(5k-3)+2(5(k+1)-4)}{2}$$

$$= \frac{5k^2-3k+10k+10-8}{2}$$

$$= \frac{5k^2+7k+2}{2}$$

$=$ RHS of $P(k+1)$ See $(*)$

ie $P(k+1)$ is true.

$\therefore$ By the Principle of Mathematical Induction $1+6+11+\ldots(5n-4) = \dfrac{n(5n-3)}{2}$ for all $n \geq 1$

8. (14 pts) Prove: For all sets A, B, and C, $\quad (A-B) \cup (C-B) \subseteq (A \cup C) - B$.

Proof. Let A, B, + C be sets.

Let $x \in (A-B) \cup (C-B)$ $\quad$ [Show $x \in (A \cup C) - B$]

Then $x \in (A-B)$ or $x \in (C-B)$

Case 1 $\quad x \in A-B$.

Then $x \in A$ and $x \notin B$.

Since $x \in A$, then $x \in A \cup C$ by def. of union

ie $x \in A \cup C$ and $x \notin B$

$\therefore$ $x \in (A \cup C) - B$ by def of set difference

Case 2 $\quad x \in (C-B)$

then $x \in C$ and $x \notin B$

Since $x \in C$, then $x \in A \cup C$ by def of union.

ie $x \in A \cup C$ and $x \notin B$

$\therefore$ $x \in (A \cup C) - B$ by def of set difference

In both cases $x \in (A \cup C) - B$

$\therefore$ $(A-B) \cup (C-B) \subseteq (A \cup C) - B$ $\blacksquare$