Triple integral ***over*** a general bounded region in 3D (i.e. a solid)

Let D be a (vertically or horizontally simple) region in the xy-plane.

Let $z = u_1(x, y)$ and $z = u_2(x, y)$ be continuous on D and $u_1(x, y) \leq u_2(x, y)$ on D.

Define the integration region E to be $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$.

[Sketch]

We want to integrate $f(x, y, z)$ over the solid region E.

Subdivide E and use approximating boxes of volume $\Delta V = \Delta x \Delta y \Delta z$

$\Rightarrow$ Triple Riemann Sum and take the limit: $\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V = \int\int\int_E f(x, y, z) dV$

Type 1 Solid Region:

D is in the $\underline{\quad xy\text{-plane}\quad}$ and the solid lies between the two surfaces $\underline{\quad z = u_1(x, y) \text{ and } z = u_2(x, y)\quad}$.

$$\int\int\int_E f(x,y,z)dV = \int\int_D \left[\int_{u_1(x,y,z)}^{u_2(x,y,z)} f(x,y,z)\ dz \right] dA$$

Furthermore:

- If D is a Type 1 Plane Region in the xy-plane [Sketch]

$$\int\int\int_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y,z)}^{u_2(x,y,z)} f(x,y,z)\ dz\ dy\ dx$$

- If D is a Type 2 Plane Region in the xy-plane [Sketch]

$$\int\int\int_E f(x,y,z)dV = \int_c^d \int_{h_1(x)}^{h_2(x)} \int_{u_1(x,y,z)}^{u_2(x,y,z)} f(x,y,z)\ dz\ dx\ dy$$

Similar for Type 2 Solid Regions:

D is in the yz-plane and the solid lies between the two surfaces $x = u_1(y, z)$ and $x = u_2(y, z)$.

Similar for Type 3 Solid Regions:

D is in the xz-plane and the solid lies between the two surfaces $y = u_1(x, z)$ and $y = u_2(x, z)$.

<u>EX</u> Evaluate $\int\int\int_E xz\ dV$ where E is the solid tetrahedron with vertices $(0,0,0), (0,1,0), (1,1,0)$, and $(0,1,1)$.

[Sketch]

<u>EX</u> $\int\int\int_E z^3 \, dV$ where E is the solid bounded by $y^2 + z^2 = 4$ and the planes $x = 0, y = 2x$, and $z = 0$ in the first octant.

<u>EX</u> Re-do the last example, but integrate with respect to x first.

Similar to $\int\int_D dA = A(D)$ $\int\int\int_E dV = V(E)$

EX Find the volume of the solid bounded by the elliptic cylinder $4x^2 + z^2 = 4$ and the planes $y = 0$ and $y = z + 2$. [Sketch]

EX Re-do the last example integrating with respect to x first.