Triple Integrals

Triple integral \underline{over} a general bounded region in 3D (i.e. <u>a solid</u>)

Let D be a (<u>vertically or horizontally simple</u>) region in the xy-plane.

Let $z = u_1(x, y)$ and $z = u_2(x, y)$ be continuous on D and $u_1(x, y) \le u_2(x, y)$ on D.

Define the <u>integration region E</u> to be $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}.$

[Sketch]

We want to integrate f(x, y, z) over the solid region E.

Subdivide E and use approximating boxes of volume $\Delta V = \Delta x \Delta y \Delta z$

 $\Rightarrow \text{ Triple Riemann Sum and take the limit: } \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*}) \Delta V = \int \int \int_{E} f(x, y, z) dV$

Type 1 Solid Region:

D is in the <u>xy-plane</u> and the solid lies between the two surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

$$\int \int \int_E f(x,y,z) dV = \int \int_D \left[\int_{u_1(x,y,z)}^{u_2(x,y,z)} f(x,y,z) dz \right] dA$$

Furthermore:

• If D is a Type 1 Plane Region in the xy-plane [Sketch]

$$\int \int \int_E f(x,y,z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y,z)}^{u_2(x,y,z)} f(x,y,z) dz \, dy \, dx$$

• If D is a Type 2 Plane Region in the xy-plane [Sketch]

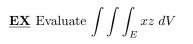
$$\int \int \int_E f(x,y,z) dV = \int_c^d \int_{h_1(x)}^{h_2(x)} \int_{u_1(x,y,z)}^{u_2(x,y,z)} f(x,y,z) dz \, dx \, dy$$

Similar for Type 2 Solid Regions:

D is in the yz-plane and the solid lies between the two surfaces $x = u_1(y, z)$ and $x = u_2(y, z)$.

Similar for Type 3 Solid Regions:

D is in the xz-plane and the solid lies between the two surfaces $y = u_1(x, z)$ and $y = u_2(x, z)$.



where E is the solid tetrahedron with vertices (0, 0, 0), (0, 1, 0), (1, 1, 0), and (0, 1, 1).

[Sketch]

 $\underline{\mathbf{EX}}$ Re-do the last example, but integrate with respect to x first.

Similar to
$$\int \int_D dA = A(D)$$
 $\int \int \int_E dV = V(E)$

<u>EX</u> Find the volume of the solid bounded by the elliptic cylinder $4x^2 + z^2 = 4$ and the planes y = 0 and y = z + 2. [Sketch]

 $\underline{\mathbf{EX}}$ Re-do the last example integrating with respect to x first.