Chain Rule for Ordinary (full/total) Derivatives

If $y = f(u)$ and $u = u(t)$, then y is ultimately a function of t and the chain rule states

$\dfrac{dy}{dt} =$ which can be rewritten as $\dfrac{dy}{dt} =$ or equivalently $\dfrac{dy}{dt} =$

Chain Rule for Partial Derivatives

Case 1 (informal) If $z = f(x, y)$ and $x = x(t)$ and $y = y(t)$, then z is ultimately a function of __________ and the chain rule states

or equivalently

Note:

- $\dfrac{dz}{dt}$ is a _______________ derivative (but the formula contains __________ derivatives).
- $\dfrac{dz}{dt}$ can be interpreted as the rate of change of z with respect to t

  as the point ____ .

  [e.g. $z =$ temperature at a point on the curve.]

EX $z = x\ln(x + 2y), \quad x = \sin t, \quad y = \cos t$

(a). Find $\dfrac{dz}{dt}$

(b). Evaluate $\dfrac{dz}{dt}$ at $t = 0$

EX Suppose $P = P(T, R)$ is the production of potatoes in cwt (hundred pounds) in Idaho in a given year where T is the average temperature and R is the annual rainfall. Suppose the average temperature is increasing at $0.05°$ F per year and the average rainfall is decreasing at a rate of 0.1 inch per year. At the current production levels it is estimated that the production will increase by 3000 cwt per degree increase in temperature and decrease 4500 cwt per inch decrease in rainfall. Estimate the current rate of change of potato production.

$$\frac{dP}{dt} =$$

Case 2 (informal) If $z = f(x, y)$ and $x = x(s, t)$ and $y = y(s, t)$, then z is ultimately a function of two variables, s and t and the chain rule states

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

s and t are ________________ variables $\Rightarrow$ x and y are ________________ variables $\Rightarrow$ z is a ________________ variable

EX Given $z = e^{xy}$ and $x = 2s - 3t$, $y = \dfrac{s}{t}$, find

(a). $\dfrac{\partial z}{\partial s}$

(b). $\dfrac{\partial z}{\partial t}$

General Case (formal) Suppose u is a ________________ function of n variables $x_1, x_2, x_3, \ldots, x_n$ and each x_j is a ________________ function of m variables $t_1, t_2, \ldots, t_m$. Then u is a ________________ function of ________________ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i} \qquad \text{for each } i = 1, 2, \ldots, m.$$

<u>EX</u> Use a tree diagram to write out the chain rule for $u = f(x, y, z)$, where $x = x(r, s), y = y(r, s)$, and $z = z(r, s)$.
$\dfrac{\partial u}{\partial r}$ is the sum of the product of partial derivatives along each path from u to r.

<u>EX</u> Given $u = xy + yz + zx$ and $x = r^2st, y = re^{st}$, and $z = rt^2$. Find $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial u}{\partial t}$ when $r = 2, s = 0$ and $t = 1$.

<u>EX</u> If $w = f(x-y, y-z, z-x)$, show that $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} + \dfrac{\partial w}{\partial z} = 0$.

Let $r = x-y, s = y-z, t = z-x$, then $w =$ ____________.

$$\frac{\partial w}{\partial x} =$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial w}{\partial r}(-1) + \frac{\partial w}{\partial s}(1) + \frac{\partial w}{\partial t}(0) = -\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z} = \frac{\partial w}{\partial r}(0) + \frac{\partial w}{\partial s}(-1) + \frac{\partial w}{\partial t}(1) = -\frac{\partial w}{\partial s} + \frac{\partial w}{\partial t}$$

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = \qquad\qquad + \left(-\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}\right) + \left(-\frac{\partial w}{\partial s} + \frac{\partial w}{\partial t}\right) = \qquad .$$

<u>EX</u> Given $y^5 + 3x^2y^2 + 5x^4 = 12$, find $\frac{dy}{dx}$ by implicit differentiation.

Use the Chain Rule for functions of more than one variable to derive a formula for the partial derivative:

Recall $F(x, y) = 0$ defines ________ implicitly as a function of ________ .

$\Rightarrow F(x, \qquad\qquad) = 0$

Implicit Differentiate using the Chain Rule:

$$\frac{\partial}{\partial x}\left[F(x, y(x))\right] = \frac{\partial}{\partial x}[0]$$

$$\frac{\partial F}{\partial x} \cdot \qquad\qquad + \frac{\partial F}{\partial y} \cdot \qquad\qquad = 0$$

$$\frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

<u>EX</u> Use this formula to find $\frac{dy}{dx}$ for $y^5 + 3x^2y^2 + 5x^4 = 12$.

Extend this idea for $z = f(x, y)$ given implicitly by $F(x, y, z(x, y)) = 0$.

$$\frac{\partial}{\partial x}\left[F(x, y, z(x, y))\right] = \frac{\partial}{\partial x}[0]$$

$$\frac{\partial z}{\partial x} = \qquad\qquad\qquad\qquad \frac{\partial z}{\partial y} =$$

EX Find the $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $\ln(x + yz) = 1 + xy^2z^3$

Higher Derivatives

EX $u = u(x, y)$ where $x = 3s + 4t$ and $y = s^2t$.

(a). Find $\dfrac{\partial u}{\partial s}$

(b). Find $\dfrac{\partial^2 u}{\partial s^2}$