Name: Key

Math 251 Multivariate Calculus – Crawford

Exam 2

26 April 2017

- You may use the provided calculator and formula sheet. But show all intermediate steps and ***leave all answers exact***.
- Other notes (in any form), books, phones, and any other unauthorized sources are not allowed.
- Clearly indicate your answers.
- ***Show all your work*** – partial credit may be given for written work.
- Unless otherwise stated, simplify/evaluate trigonometric, exponential, logarithmic, and hyperbolic functions for standard values.
- Good Luck!

Calculator Number: ____________

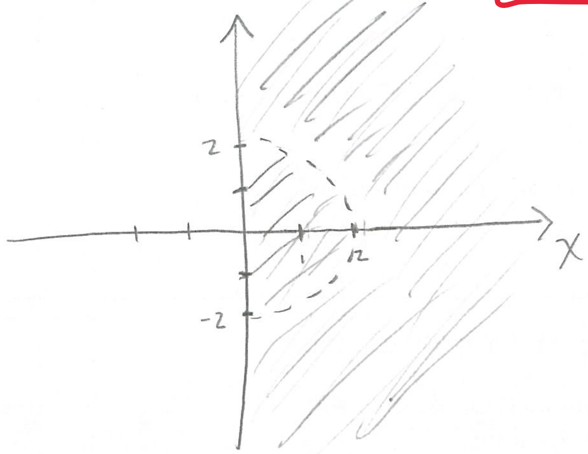
Score

1	/6
2	/10
3	/14
4	/14
5	/14
6	/18
7	/10
8	/8
9	/8
10	/2
Total	/100

1. (6 pts). Find and sketch the domain of $f(x, y) = \dfrac{\sqrt{x}}{4 - x^2 - y^2}$

$\boxed{x \geq 0}$ and $4 - x^2 - y^2 \neq 0$

$\Rightarrow \boxed{x^2 + y^2 \neq 4}$



2. (10 pts). Find the limit, if it exists, or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$$

Let $y = mx$:

$$\Rightarrow \lim_{x\to 0} \frac{x^4 - 4(mx)^2}{x^2 + 2(mx)^2}$$

$$= \lim_{x\to 0} \frac{x^4 - 4m^2x^2}{x^2 + 2m^2x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}(x^2 - 4m^2)}{\cancel{x^2}(1 + 2m^2)}$$

$$= \lim_{x\to 0} \frac{x^2 - 4m^2}{1 + 2m^2}$$

$$= \underbrace{\frac{-4m^2}{1 + 2m^2}}$$

Limit depends on the slope m of the line

$\therefore$ $\boxed{\text{Limit DNE}}$

3. (14 pts). For each of the given functions, find the indicated partial derivative. [You do not need to simplify.]

(a). Find f_{yyx} for $f(x,y) = xy^4e^{-x}$.

$f_y = 4xy^3e^{-x}$

$f_{yy} = 12xy^2e^{-x} = 12y^2xe^{-x}$

$\boxed{f_{yyx} = 12y^2x \cdot e^{-x}(-1) + e^{-x}(12y^2)}$

$= -12xy^2e^{-x} + 12y^2e^{-x}$

(b). Find $\dfrac{\partial z}{\partial y}$ where $z = z(x,y)$ is defined implicitly by $yz + x\ln y = z^2$.

$\underbrace{yz + x\ln y - z^2}_{F(x,y,z)} = 0$

$F(x,y,z) = 0$

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = \boxed{-\dfrac{z + x\cdot\frac{1}{y}}{y - 2z}}$

OR $\dfrac{\partial}{\partial y}[yz + x\ln y] = \dfrac{\partial}{\partial y}[z^2]$

$y\cdot\dfrac{\partial z}{\partial y} + z\cdot 1 + x\cdot\dfrac{1}{y} = 2z\dfrac{\partial z}{\partial y}$

$y\dfrac{\partial z}{\partial y} - 2z\dfrac{\partial z}{\partial y} = -\dfrac{x}{y} - z$

$\dfrac{\partial z}{\partial y}(y - 2z) = -\left(\dfrac{x}{y} + z\right)$

$\boxed{\dfrac{\partial z}{\partial y} = \dfrac{-\left(\frac{x}{y} + z\right)}{y - 2z}}$

✓

4. (14 pts). Given $f(x,y) = \frac{x}{y^2}$.

(a). Find an equation of the tangent plane to the surface $z = f(x,y)$ at the point $(-4, 2)$.

① pt $f(-4,2) = \frac{-4}{(2)^2} = -1$ ie pt $(-4,2,-1)$

② $f_x = \frac{1}{y^2}\Big|_{(-4,2)} = \frac{1}{(2)^2} = \frac{1}{4}$

$f_y = \frac{-2x}{y^3}\Big|_{(-4,2)} = \frac{-2(-4)}{(2)^3} = \frac{8}{8} = 1$

$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$$\boxed{z + 1 = \frac{1}{4}(x+4) + 1(y-2)}$$

(b). Find the linearization $L(x,y)$ of f at the point $(-4,2)$.

Same as tangent plane in form $z = L(x,y)$

$$\boxed{L(x,y) = -1 + \frac{1}{4}(x+4) + (y-2)}$$

(c). Use your answer from part (b) to approximate $f(-3.9, 2.2)$.

$f(-3.9, 2.2) \approx L(-3.9, 2.2) = -1 + \frac{1}{4}(-3.9+4) + (2.2-2)$

$= -1 + \frac{1}{4}(0.1) + (0.2)$

$= \boxed{-0.775}$

5. (14 pts). Suppose that over a certain region of space the electrical potential V is given by

$V(x, y, z) = 6x^2 - 4xy + 2xyz$,

(a). Find the rate of change of the potential V at $P(-2, -1, 3)$ in the direction of $\langle 1, 3, -1\rangle$.

$$\vec{u} = \frac{\langle 1, 3, -1\rangle}{\sqrt{1+9+1}} = \frac{\langle 1, 3, -1\rangle}{\sqrt{11}} = \left\langle \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}} \right\rangle$$

$$\nabla V = \langle 12x - 4y + 2yz, -4x + 2xz, 2xy \rangle$$

$$\nabla V \Big|_{(-2,-1,3)} = \langle 12(-2) - 4(-1) + 2(-1)(3), -4(-2) + 2(-2)(3), 2(-2)(-1) \rangle$$

$$= \langle -26, -4, 4 \rangle$$

$$D_{\vec{u}} V = \nabla V \cdot \vec{u} = \frac{-26(1) - 4(3) + 4(-1)}{\sqrt{11}} = \boxed{\frac{-42}{\sqrt{11}}}$$

(b). In which direction is V decreasing the fastest at $P(-2, -1, 3)$?

$$-\nabla V = \boxed{\langle 26, 4, -4 \rangle}$$

6. (18 pts). Find all the critical points of f and determine if each yields a local maximum or minimum or is a saddle point. In the case of a maximum or minimum, find the local maximum or minimum value(s).

$f(x, y) = x^3 - 3x + \frac{1}{3}xy^2$,

$f_x = 3x^2 - 3 + \frac{1}{3}y^2 = 0$ AND $f_y = \frac{2}{3}xy = 0$

$\Rightarrow x = 0$ or $y = 0$

Case 1 $x = 0$:

$-3 + \frac{1}{3}y^2 = 0$

$\frac{1}{3}y^2 = 3$

$y^2 = 9$

$y = \pm 3$

ie Crit pts: $(0, \pm 3)$

Case 2 $y = 0$:

$3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

ie Crit pts $(\pm 1, 0)$

$f_{xx} = 6x$

$f_{yy} = \frac{2}{3}x$

$f_{xy} = \frac{2}{3}y$

$D = f_{xx}f_{yy} - (f_{xy})^2$

$D = (6x)(\frac{2}{3}x) - (\frac{2}{3}y)^2$

$D(0,3) = 0 - (\frac{2}{3}\cdot 3)^2 = -4 < 0$

$\Rightarrow$ $\boxed{(0,3) \text{ Saddle}}$

$D(0,-3) = 0 - (\frac{2}{3}(-3))^2 = -4 < 0$

$\Rightarrow$ $\boxed{(0,-3) \text{ Saddle}}$

$D(1,0) = (6)(\frac{2}{3}) - (0)^2 = 4 > 0$

and $f_{xx} = 6 > 0$

$\boxed{\text{Local min @ } (1,0)}$

$f(1,0) = (1)^3 - 3(1) + \frac{1}{3}(1)(0)^2$

$= \boxed{-2}$ ← Local min Value

$D(-1,0) = (-6)(-\frac{2}{3}) - 0^2 = 4 > 0$ and $f_{xx} = -6 < 0$

$\boxed{\text{Local max @ } (-1,0)}$

$f(-1,0) = (-1)^3 - 3(-1) + \frac{1}{3}(-1)(0)^2 = \boxed{2}$ ← Local Max Value

7. (10 pts). Evaluate the following integral. [Show all work. Leave your answer exact and simplify.]

$$\int_0^{\sqrt[3]{\pi}} \int_0^{x^2} y^2 + \sin(x^3)\, dy\, dx = \int_0^{\sqrt[3]{\pi}} \left[\frac{1}{3}y^3 + y\sin(x^3)\right]_0^{x^2} dx$$

$$= \int_0^{\sqrt[3]{\pi}} \left[\frac{1}{3}(x^2)^3 + x^2\sin(x^3) - (0-0)\right] dx$$

$$= \int_0^{\sqrt[3]{\pi}} \frac{1}{3}x^6 + x^2\sin(x^3)\, dx$$

$u = x^3$

$du = 3x^2dx$

$\frac{1}{3}du = x^2dx$

$$= \frac{1}{21}x^7 + \frac{1}{3}\int_{x=0}^{x=\sqrt[3]{\pi}} \sin(u)\, du$$

$$= \frac{1}{21}x^7 - \frac{1}{3}\cos(u)\Big|_{x=0}^{x=\sqrt[3]{\pi}}$$

$$= \frac{1}{21}x^7 - \frac{1}{3}\cos(x^3)\Big|_0^{\sqrt[3]{\pi}}$$

$$= \frac{1}{21}(\sqrt[3]{\pi})^7 - \frac{1}{3}\cos(\pi) - \left(\frac{1}{21}\cdot 0 - \frac{1}{3}\cos(0)\right)$$

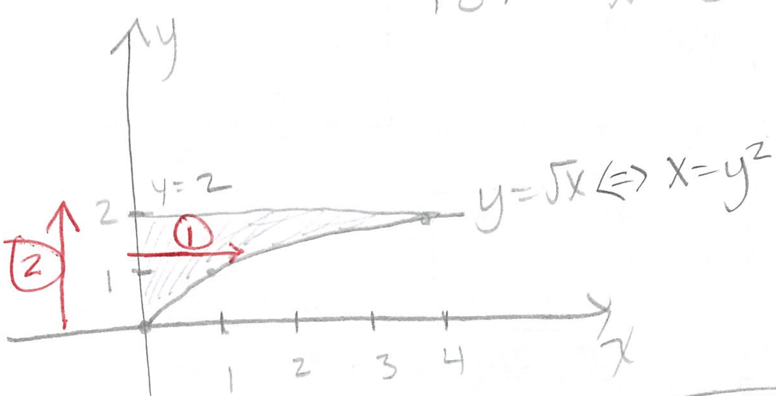
$$= (\pi)^{7/3}/21 - \frac{1}{3}(-1) + \frac{1}{3}(1) = \boxed{\frac{(\pi)^{7/3}}{21} + \frac{2}{3}}$$

8. (8 pts). Sketch the region of integration and change the order. [Do not evaluate.]

$$\int_0^4 \int_{\sqrt{x}}^2 x^2y\, dy\, dx$$

$y = \sqrt{x}$ to $y = 2$

for $x = 0$ to 4



Reverse:

$x = 0$ to $x = y^2$

for $y = 0$ to 2

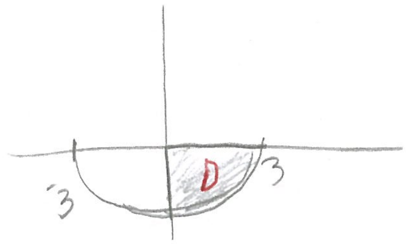
$$\boxed{\int_0^2 \int_0^{y^2} x^2y\, dx\, dy}$$

9. (8 pts). Convert the following integral to polar coordinates. [Do not evaluate.]

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{0} \frac{y^2}{x^2+y^2}\, dy\, dx$$

D: $y = -\sqrt{9-x^2}$ to $y = 0$ for $x = 0$ to 3

Bottom ½ circle $x^2+y^2=9$



D: $r = 0$ to $r = 3$ for $\theta = \frac{3\pi}{2}$ to 2π

$$\boxed{\int_{3\pi/2}^{2\pi} \int_0^3 \frac{r^2 \sin^2\theta}{r^2} \cdot r\, dr\, d\theta}$$

$$= \int_{3\pi/2}^{2\pi} \int_0^{\pi} r \sin^2\theta \, dr\, d\theta$$

10. (2 pts). Multiple Choice

f is ultimately a fcn of s & t

Given that $f = f(x, y, z)$ where $x = x(s, t)$, $y = y(s, t)$, and $z = z(s, t)$, circle the correct form of the chain rule.

(a). $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial z}$, $\quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial x} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial y} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial z}$

(b). $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial x}$, $\quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial y}$, $\quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial z}$

(c). $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}$, $\quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$