All Derivative Formulas MUST come from the limit definition of the derivative!

Fill in the spaces/blanks below to prove each of the formulas.

1. $f(x) = c$

[Sketch picture]

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$

$\Longrightarrow$ Constant Rule: If $f(x) = c$, then $f'(x) =$ 0

2. $f(x) = x$

[Sketch picture]

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$

$\Longrightarrow$ x Rule: If $f(x) = x$, then $f'(x) =$ 1

3. $f(x) = cx$

[Sketch picture]

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{c(x+h) - cx}{h} = \lim_{h\to 0} \frac{cx + ch - cx}{h} = \lim_{h\to 0} \frac{ch}{h} = \lim_{h\to 0} c = c$

$\Longrightarrow$ cx Rule: If $f(x) = cx$, then $f'(x) =$ c

Expansions used in #4-7: $(x+h)^2 = x^2 + 2xh + h^2$ $\quad$ $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ $\quad$ $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ $\quad$ $(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$

4. Find $f'(x)$ for $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 \quad - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2 \quad - x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x+0 = 2x \end{aligned}$$

5. Find $f'(x)$ for $f(x) = x^3$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 \quad - x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2+3x(0)+(0)^2 = 3x^2 \end{aligned}$$

6. Find $f'(x)$ for $f(x) = x^4$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 \quad - x^4}{h} = \lim_{h\to 0} \frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h} \\ &= \lim_{h\to 0} \frac{x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0} \frac{h(4x^3+6x^2h+4xh^2+h^3)}{h} = \lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3) \\ &= 4x^3+6x^2(0)+4x(0)^2+(0)^3 = 4x^3 \end{aligned}$$

7. Find $f'(x)$ for $f(x) = x^5$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^5 \quad - x^5}{h} = \lim_{h\to 0} \frac{x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5-x^5}{h} \\ &= \lim_{h\to 0} \frac{5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5}{h} = \lim_{h\to 0} \frac{h(5x^4+10x^3h+10x^2h^2+5xh^3+h^4)}{h} \\ &= \lim_{h\to 0}(5x^4+10x^3h+10x^2h^2+5xh^3+h^4) = 5x^4+10x^3(0)+10x^2(0)^2+5x(0)^3+(0)^4 = 5x^4 \end{aligned}$$

Do you see a pattern? Guess the formula for the derivative of $f(x) = x^n$:

Power Rule: If $f(x) = x^n$, then $f'(x) =$ nx^{n-1}