Let's expand the idea from the last worksheet to find a power series $\sum_{n=0}^{\infty} c_n x^n$ to represent a generic function $f(x)$

In other words, given $f(x)$ find the coefficients $c_0, c_1, c_2, \ldots$ so that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

Start by differentiating the above expression. That is, find the derivatives of $f(x)$ and differentiate the power series. Then evaluate them both at at $x = 0$.

$$\begin{array}{rcl} f(x) &=& c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\ f'(x) &=& c_1 + 2\,c_2 x + 3\,c_3 x^2 + 4\,c_4 x^3 + 5\,c_5 x^4 + \cdots \\ f''(x) &=& 2\,c_2 + 2\cdot 3\,c_3 x + 3\cdot 4\,c_4 x^2 + 4\cdot 5\,c_5 x^3 + \cdots \\ f'''(x) &=& 2\cdot 3\,c_3 + 2\cdot 3\cdot 4\,c_4 x + 3\cdot 4\cdot 5\,c_5 x^2 + \cdots \\ f^{(iv)}(x) &=& 2\cdot 3\cdot 4\,c_4 + 2\cdot 3\cdot 4\cdot 5\,c_5 x + \cdots \\ \vdots && \\ f^{(n)}(x) &=& \end{array} \qquad \begin{array}{rcl} f(0) &=& c_0 \\ f'(0) &=& \\ f''(0) &=& \\ f'''(0) &=& \\ f^{(iv)}(0) &=& \\ && \\ f^{(n)}(0) &=& \end{array}$$

Remember $f(x)$ is given (i.e. known) and hence

$f(0), f'(0), f''(0), \ldots, f^{(n)}(0)$ ____ .

So the only unknown in equation $f^{(n)}(0) = n!c_n$ is ________ .

Lucky for us, the coefficients c_n are exactly what we set out to find. Solve:

Repeat the process for a power series centered at $x = a$.

In other words, given $f(x)$ find the coefficients $c_0, c_1, c_2, \ldots$ so that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \cdots$$

Start by finding the derivatives and evaluating them at $x = a$.

$$\begin{aligned}
f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \cdots & f(a) &= c_0 \\
f'(x) &= c_1 + 2\,c_2(x-a) + 3\,c_3(x-a)^2 + 4\,c_4(x-a)^3 + 5\,c_5(x-a)^4 + \cdots & f'(a) &= \\
f''(x) &= 2\,c_2 + 2\cdot 3\,c_3(x-a) + 3\cdot 4\,c_4(x-a)^2 + 4\cdot 5\,c_5(x-a)^3 + \cdots & f''(a) &= \\
f'''(x) &= 2\cdot 3\,c_3 + 2\cdot 3\cdot 4\,c_4(x-a) + 3\cdot 4\cdot 5\,c_5(x-a)^2 + \cdots & f'''(a) &= \\
f^{(iv)}(x) &= 2\cdot 3\cdot 4\,c_4 + 2\cdot 3\cdot 4\cdot 5\,c_5(x-a) + \cdots & f^{(iv)}(a) &= \\
&\vdots \\
f^{(n)}(x) &= & f^{(n)}(a) &=
\end{aligned}$$

Remember $f(x)$ and a are given (i.e. known) and hence

$f(a), f'(a), f''(a), \ldots, f^{(n)}(a)$ ____ .

So the only unknown in equation $f^{(n)}(a) = n!c_n$ is ________ .

Lucky for us, the coefficients c_n are exactly what we set out to find. Solve: