Take-Home Quiz 2 Key

1. $\int \frac{1}{x\sqrt{4x^2+1}}\,dx$ $x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\, d\theta$

$= \int \frac{1}{\frac{1}{2}\tan\theta\sqrt{4\cdot\frac{1}{4}\tan^2\theta+1}}\cdot\frac{1}{2}\sec^2\theta\,d\theta = \int\frac{\sec^2\theta}{\tan\theta\sqrt{\tan^2\theta+1}}\,d\theta$

$= \int\frac{\sec^2\theta}{\tan\theta\sqrt{\sec^2\theta}}\,d\theta = \int\frac{\sec^2\theta}{\tan\theta\sec\theta}\,d\theta = \int\frac{\sec\theta}{\tan\theta}\,d\theta$

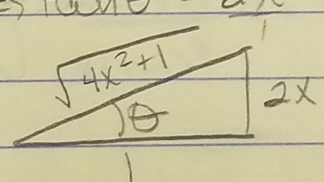
$= \int\frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)}\,d\theta = \int\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta = \int\frac{1}{\sin\theta}\,d\theta$

$= \int\csc\theta\,d\theta = -\ln|\csc\theta+\cot\theta|+C$

From subs:

$x = \frac{1}{2}\tan\theta$

$\Rightarrow \tan\theta = 2x$



$(1)^2+(2x)^2=c^2$

$1+4x^2=c^2$

$$\boxed{\begin{aligned} &= -\ln\left|\frac{\sqrt{4x^2+1}}{2x}+\frac{1}{2x}\right|+C \\ &= -\ln\left|\frac{\sqrt{4x^2+1}+1}{2x}\right|+C \end{aligned}}$$

2. $\int \frac{x+\arcsin(x)}{\sqrt{1-x^2}}\,dx$

$= \int \frac{x}{\sqrt{1-x^2}} + \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

$= \underbrace{\int\frac{x}{\sqrt{1-x^2}}\,dx} + \underbrace{\int\frac{\arcsin x}{\sqrt{1-x^2}}\,dx}$

$w = 1-x^2$ $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}$

$dw = -2x\,dx \Rightarrow -\frac{1}{2}dw = x\,dx$

$\Rightarrow -\frac{1}{2}\int\frac{1}{\sqrt{w}}\,dw + \int u\,du$

$= -\frac{1}{2}\int w^{-1/2}\,dw + \int u\,du$

$= -\frac{1}{2}\cdot 2\cdot w^{1/2} + \frac{1}{2}u^2 + C$

$= \boxed{-\sqrt{1-x^2} + \frac{1}{2}(\arcsin x)^2 + C}$

same

OR

$\int \frac{x+\arcsin(x)}{\sqrt{1-x^2}}\,dx$

$\underbrace{u = \arcsin x} \Rightarrow du = \frac{1}{\sqrt{1-x^2}}\,dx$

$\Rightarrow \sin u = x$

subs $\Rightarrow$

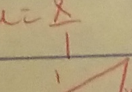
$\int(\sin u + u)\,du$

$= -\cos u + \frac{1}{2}u^2 + C$

$= \boxed{-\cos(\arcsin x) + \frac{1}{2}(\arcsin x)^2 + C}$

Note: $u = \arcsin x$

$\sin u = \frac{x}{1}$



$\Rightarrow$

$-\cos(u)$

$= -\frac{\sqrt{1-x^2}}{1}$

same

$= -\sqrt{1-x^2}$

3. $\displaystyle\int \frac{x^3-4x-1}{x(x-1)^3}\,dx$ $\qquad \displaystyle\frac{x^3-4x-1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$

$$= \frac{A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx}{x(x-1)^3}$$

$$\Rightarrow x^3-4x-1 = A(x^3-3x^2+3x-1) + Bx(x^2-2x+1) + Cx^2 - Cx + Dx$$

$$= Ax^3 - 3Ax^2 + 3Ax - A + Bx^3 - 2Bx^2 + Bx + Cx^2 - Cx + Dx$$

$$= (A+B)x^3 + (-3A-2B+C)x^2 + (3A+B-C+D)x - A$$

LHS = RHS

$x^3:\ 1 = A+B$ $\qquad 1 = 1 + B \Rightarrow \boxed{B = 0}$ (2)

$x^2:\ 0 = -3A-2B+C$ $\qquad \Rightarrow 0 = -3(1) - 2(0) + C$

$x:\ -4 = 3A+B-C+D$ $\qquad \boxed{3 = C}$ (3)

const: $-1 = -A$ $\Rightarrow$ (1) $\boxed{A = 1}$ $\qquad \Rightarrow -4 = 3(1) + 0 - 3 + D$

$\boxed{-4 = D}$ (4)

$$\int \frac{x^3-4x-1}{x(x-1)^3}\,dx = \int \frac{1}{x} + \underbrace{\frac{3}{(x-1)^2} - \frac{4}{(x-1)^3}}\,dx$$

$u = x-1 \Rightarrow du = dx$

$$= \ln|x| + \int (3u^{-2} - 4u^{-3})\,du$$

$$= \ln|x| + 3\frac{u^{-1}}{-1} - 4\frac{u^{-2}}{-2} + C$$

$$= \ln|x| - \frac{3}{u} + \frac{2}{u^2} + C$$

$$= \boxed{\ln|x| - \frac{3}{x-1} + \frac{2}{(x-1)^2} + C}$$

4. $\displaystyle\int x^3 \ln x\,dx$ $\qquad u = \ln x \quad dv = x^3$

$\qquad du = \frac{1}{x}\,dx \xleftarrow{\int} v = \frac{1}{4}x^4$

$$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$$

$$= \boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$$