Name: Key

Math 152, Calculus II – Crawford

Exam 3

29 April 2018

	Score
1	/10
2	/12
3	/30
4	/12
5	/24
6	/10
7	/6
Total	/100

- Calculators, books, or notes (in any form) are not allowed. Having a phone out will result in an automatic 0 grade.
- You may use the given formula sheet.
- Clearly indicate your answers.
- ***Show all your work*** – partial credit may be given for written work.
- Good Luck!

1. (10 pts). Determine whether the following series converges or diverges. If it converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{4(-2)^{n-2}}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{4(-2)^n(-2)^{-2}}{3^n \cdot 3} = \sum_{n=1}^{\infty} \frac{4(-2)^n}{3^n \cdot 3 \cdot (-2)^2}$$

$$= \sum_{n=1}^{\infty} \frac{\cancel{4}(-2)^n}{3^n \cdot 3 \cdot \cancel{4}} = \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(-\frac{2}{3}\right)^n = \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1-\left(-\frac{2}{3}\right)}$$

$|r| = \left|-\frac{2}{3}\right| < 1$

$\therefore$ converging geom. series

$$= \frac{\left(-\frac{2}{9}\right)}{1+\frac{2}{3}} = \frac{\left(-\frac{2}{9}\right)}{\left(\frac{5}{3}\right)} = -\frac{2}{\cancel{9}_{3}} \cdot \frac{\cancel{3}}{5}$$

$$= \boxed{-\frac{2}{15}}$$

2. (12 pts). Given $\sum_{n=1}^{\infty} ne^{-n^2}$.

(a). Use the integral test to show that it converges. [You do not need to show that it decreases.]

$$\int_1^{\infty} xe^{-x^2}\,dx = \int_{x=1}^{x=\infty} -\frac{1}{2}e^u\,du = -\frac{1}{2}e^u \Big|_{x=1}^{x=\infty}$$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$= -\frac{1}{2}e^{-x^2}\Big|_1^{\infty}$$

$$= \lim_{x\to\infty}\left(-\frac{1}{2}e^{-x^2}\right) + \frac{1}{2}e^{-1}$$

0

$$= \frac{1}{2}e^{-1} \leftarrow \text{Integral converges.}$$

$\therefore$ $\boxed{\text{Series } \sum_{n=1}^{\infty} ne^{-n^2} \text{ converges by the Integral Test}}$

(b). If s_{10} is used to approximate the infinite series, what is the bound for the maximum possible error? (i.e. bound on $|R_{10}|$?) [Leave your answer exact.]

$$|R_{10}| \le \int_{10}^{\infty} xe^{-x^2}\,dx$$

$$= -\frac{1}{2}e^{-x^2}\Big|_{10}^{\infty}$$

$$= \lim_{x\to\infty}\left(-\frac{1}{2}e^{-x^2}\right) + \frac{1}{2}e^{-10^2}$$

0

$$= \frac{1}{2}e^{-100}$$

ie $\boxed{|R_{10}| \le \frac{1}{2}e^{-100}}$

3. (30 pts). Determine whether the following series converge or diverge. [Show all your work and clearly indicate any tests that you use.]

(a). $\sum_{n=1}^{\infty} \left(\frac{n-8n^2}{3n^2+2}\right)^n$

Root: $\lim_{n\to\infty} \left| \left(\frac{n-8n^2}{3n^2+2}\right)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{n-8n^2}{3n^2+2} \right|$

$\rho = \left| -\frac{8}{3} \right| = \boxed{\frac{8}{3} > 1 \quad \therefore \sum_{n=1}^{\infty} \left(\frac{n-8n^2}{3n^2+2}\right)^n \text{ diverges by the Root Test}}$

(b). $\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2+4}$

L.O. $\frac{1}{n^2}$

$\sin n \leq 1$ and $n^2+4 > n^2$

$\Rightarrow \underbrace{2+\sin n \leq 2+1 = 3}$ $\underbrace{\frac{1}{n^2+4} < \frac{1}{n^2}}$

combine

$\sum \frac{1}{n^2}$ converges

$\frac{2+\sin n}{n^2+4} < \frac{3}{n^2}$ and since $\sum \frac{3}{n^2} = 3\sum \frac{1}{n^2}$ is a converging p-series, then

$\boxed{\sum \frac{2+\sin n}{n^2+4} \text{ converges by the Comparison test.}}$

(c). $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n10^n}$

Ratio: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)!}{(n+1)10^{n+1}} \cdot \frac{n\,10^n}{(-1)^n n!} \right|$

$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^n}(-1)^1 \cancel{n!}(n+1)}{(n+1)\cdot \cancel{10^n}\,10} \cdot \frac{n\cdot \cancel{10^n}}{\cancel{(-1)^n}\cancel{n!}} \right|$

$= \lim_{n\to\infty} \left| \frac{(-1)\cancel{(n+1)}(n)}{\cancel{(n+1)}\,10} \right| = \lim_{n\to\infty} \left| \frac{-n}{10} \right| = \infty > 1$

$\therefore$ Series diverges by the Ratio test.

4. (12 pts). Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. [Show all your work and clearly indicate any tests that you use.]

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{n^2+n}$$

oops... should have been $\frac{3n}{n^2+1}$

① Consider $\sum \left| \frac{(-1)^n 3n}{n^2+n} \right| = \sum \frac{3n}{n^2+n} = \sum \frac{3}{n+1}$

L.O. $\frac{1}{n} \Rightarrow \lim_{n\to\infty} \frac{\left(\frac{3}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{3}{n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{3n}{n+1} = 3$

which is nonzero & finite. Then since $\sum \frac{1}{n}$ is a diverging p-series, then $\sum \frac{3n}{n^2+n}$ diverges by the Limit Comp. Tes.

② $\sum (-1)^n \frac{3n}{n^2+n}$

① $\lim_{n\to\infty} \frac{3n}{n^2+n} = 0$

② $\frac{3n}{n^2+n} = \frac{3}{n+1}$ is clearly decreasing

or $f(x) = \frac{3x}{x^2+x} \Rightarrow f'(x) = \frac{(x^2+x)(3) - 3x(2x+1)}{(x^2+x)^2}$

$= \frac{3x^2+3x-6x^2-3x}{(x^2+x)^2}$

$= \frac{-3x^2}{(x^2+x)^2} < 0$ for all x $(\neq 0)$

$\therefore$ decreasing

$\therefore \sum (-1)^n \frac{3n}{n^2+n}$ converges by the Alternating Series Test.

Since $\sum \frac{(-1)^n 3n}{n^2+n}$ converges, but $\sum \left|\frac{(-1)^n 3n}{n^2+n}\right| = \sum \frac{3n}{n^2+n}$ diverges,

$\Rightarrow$ $\boxed{\sum \frac{(-1)^n 3n}{n^2+n} \text{ converges conditionally}}$

5. (24 pts). Find the radius of convergence and *exact* interval of convergence for the following series.

(a). $\sum_{n=1}^{\infty} \frac{n^2(x+1)^n}{2^n}$

Ratio: $\lim_{n\to\infty} \left| \frac{(n+1)^2 (x+1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2(x+1)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2 \cancel{(x+1)^n}(x+1)}{\cancel{2^n}\cdot 2} \cdot \frac{\cancel{2^n}}{n^2\cancel{(x+1)^n}} \right|$

$= \lim_{n\to\infty} \left| \frac{(x+1)}{2} \cdot \frac{(n+1)^2}{n^2} \right| = \lim_{n\to\infty} \left| \frac{(x+1)}{2} \cdot \frac{n^2+2n+1}{n^2} \right|$ $\to 1$

$= \frac{|x+1|}{2} < 1$ for convergence

$\Rightarrow |x+1| < 2 \Rightarrow \boxed{R = 2}$

$\Rightarrow -2 < x+1 < 2$

$-3 < x < 1$ Go to (*)

(*) Endpts

$x=-3$: $\sum \frac{n^2(-3+1)^n}{2^n}$

$= \sum \frac{n^2(-2)^n}{2^n} = \sum n^2(-1)^n$

Diverges by Test for Div.

$x=1$: $\sum \frac{n^2(1+1)^n}{2^n} = \sum \frac{n^2 \cancel{2^n}}{\cancel{2^n}}$

$= \sum n^2$ Diverges by Test for Div.

$\Rightarrow \boxed{\text{I of C: } (-3,1)}$

(b). $\sum_{n=1}^{\infty} \frac{n!}{n+1} x^{2n}$

Ratio: $\lim_{n\to\infty} \left| \frac{(n+1)!}{n+1+1} x^{2(n+1)} \cdot \frac{n+1}{n!\, x^{2n}} \right|$

$= \lim_{n\to\infty} \left| \frac{\cancel{n!}(n+1)\cancel{x^{2n}}x^2}{n+2} \cdot \frac{n+1}{\cancel{n!}\,\cancel{x^{2n}}} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n+2} \cdot x^2 \right| = \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n+2} \cdot x^2 \right| = \infty > 1$

for all $x \neq 0$

centered at $x=0$

$\Rightarrow$ Diverges for all $x \neq 0$.

ie $\boxed{R = 0 \qquad \text{I of C: pt. } x=0}$

6. (10 pts). Use a known power series to find a power series representation for the function $f(x) = \dfrac{x}{(1+4x)^2}$.
[Simplify your answer, but do not shift any indices.]

$$f(x) = x \cdot \frac{1}{(1-(-4x))^2} = x \cdot \sum_{n=0}^{\infty} (n+1)(-4x)^n$$

multiply.

$$= x \cdot \sum_{n=0}^{\infty} (n+1)(-1)^n 4^n x^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n (n+1) 4^n x^{n+1}}$$

7. (6 pts). TRUE OR FALSE. Determine whether the following statement is true or false.

(a). T F If $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{\pi^2 - 6}{6}$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

$$\checkmark\ \frac{\pi^2 - 6}{6} = \frac{\pi^2}{6} - 1 = \sum_{n=2}^{\infty} \frac{1}{n^2}$$

(b). T F If $\displaystyle\sum_{n=1}^{\infty} c_n x^n$ converges for $x = -3$. but diverges for $x = 4$. then it must converge for $x = 3$.

Not enough info to determine

(c). T F If $\displaystyle\sum_{n=1}^{\infty} c_n x^n$ converges for $x = -3$. but diverges for $x = 4$. then it must diverge for $x = -5$.