1. (16 pts). Evaluate the following limits. Clearly indicate all steps.

(a). $\lim_{x\to 0} \dfrac{x^2}{1-\cos x}$ $\overset{L}{=} \lim_{x\to 0} \dfrac{2x}{\sin x} \overset{L}{=} \lim_{x\to 0} \dfrac{2}{\cos x} = \dfrac{2}{\cos(0)}$

$\dfrac{0}{1-1} \to \dfrac{0}{0}$ $\qquad \dfrac{0}{0}$ $\qquad = \dfrac{2}{1} = \boxed{2}$

(b). $\lim_{x\to 1^+} x^{2/(1-x)}$

$1^{2/0} \to 1^{\infty}$

Ind. Form

Let $L = \lim_{x\to 1^+} x^{\frac{2}{1-x}}$

$\ln L = \lim_{x\to 1^+} \ln x^{\frac{2}{1-x}}$

$= \lim_{x\to 1^+} \dfrac{2}{1-x} \cdot \ln x$

$-\infty \cdot 0$

$= \lim_{x\to 1^+} \dfrac{2\ln x}{1-x}$ $\quad \dfrac{0}{0}$

$\overset{L}{=} \lim_{x\to 1^+} \dfrac{2\cdot\frac{1}{x}}{-1}$

$= \lim_{x\to 1^+} \dfrac{-2}{x}$

$= -2$

ie $\ln L = -2$

$L = \boxed{e^{-2}}$ ie $\lim_{x\to 1^+} x^{\frac{2}{1-x}} = e^{-2}$

2. (10 pts). Use Simpson's Rule with $n = 8$ to **set up** the approximation for the integral $\int_2^4 \frac{1}{\ln x}\,dx$.

[Do not simplify!!] $\Delta x = \dfrac{4-2}{8} = \dfrac{2}{8} = \dfrac{1}{4} = 0.25$

2 2.25 2.5 2.75 3 3.25 3.5 3.75 4

x_0 x_1 . . x_8

$S_8 = \frac{1}{3}\Delta x \left[f(2) + 4f(2.25) + 2f(2.5) + 4f(2.75) + 2f(3) + 4f(3.25) + 2f(3.5) + 4f(3.75) + f(4) \right]$

$$\boxed{S_8 = \frac{1}{3}\left(\frac{1}{4}\right)\left[\frac{1}{\ln 2} + \frac{4}{\ln 2.25} + \frac{2}{\ln 2.5} + \frac{4}{\ln 2.75} + \frac{2}{\ln 3} + \frac{4}{\ln 3.25} + \frac{2}{\ln 3.5} + \frac{4}{\ln 3.75} + \frac{1}{\ln 4} \right]}$$

3. (38 pts). Evaluate the following integrals. [Part (c) is on next page.]

(a). $\int \sin^3 x \cos^5 x \, dx$

$u = \sin x$
$du = \cos x dx$

$= \int \sin^3 x \cos^4 x \cos x dx$

$= \int \sin^3 x (\cos^2 x)^2 \cos x dx$

$= \int \sin^3 x (1-\sin^2 x)^2 \cos x dx$

$= \int u^3 (1-u^2)^2 du$

$= \int u^3 (1-2u^2+u^4) du$

$= \int u^3 - 2u^5 + u^7 du$

$= \frac{1}{4}u^4 - \frac{2}{6}u^6 + \frac{1}{8}u^8 + C$

$= \boxed{\frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x + C}$

OR

$u = \cos x$
$du = -\sin x dx$
$-du = \sin x dx$

$\int \sin^3 x \cos^5 x \, dx$

$= \int \sin^2 x \cos^5 x \sin x dx$

$= \int (1-\cos^2 x) \cos^5 x \sin x dx$

$= -\int (1-u^2) u^5 du$

$= -\int u^5 - u^7 du$

$= \int u^7 - u^5 du$

$= \frac{1}{8}u^8 - \frac{1}{6}u^6 + C$

$= \boxed{\frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C}$

(b). $\int (x^3 - 2x) \ln x \, dx$

$u = \ln x \qquad dv = (x^3 - 2x) dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{4}x^4 - x^2$

$= \left(\frac{1}{4}x^4 - x^2\right)\ln x - \int \left(\frac{1}{4}x^4 - x^2\right)\frac{1}{x} dx$

$= \left(\frac{1}{4}x^4 - x^2\right)\ln x - \int \frac{1}{4}x^3 - x \, dx$

$= \boxed{\left(\frac{1}{4}x^4 - x^2\right)\ln x - \left(\frac{1}{16}x^4 - \frac{1}{2}x^2\right) + C}$

$= \boxed{\left(\frac{1}{4}x^4 - x^2\right)\ln x - \frac{1}{16}x^4 + \frac{1}{2}x^2 + C}$

(c). $\int \frac{\sqrt{x^2-9}}{x}\,dx$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sqrt{9\sec^2\theta - 9}}{\cancel{3\sec\theta}}\,\cancel{3\sec\theta}\tan\theta\,d\theta$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

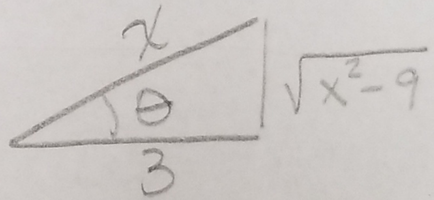
$$= \int \sqrt{9(\sec^2\theta - 1)}\,\tan\theta\,d\theta$$

$$= \int \sqrt{9\tan^2\theta}\,\tan\theta\,d\theta$$

$$= \int 3\tan\theta\tan\theta\,d\theta$$

$\sec\theta = \frac{x}{3} = \frac{hyp}{adj}$

$$= \int 3\tan^2\theta\,d\theta$$



$$= \int 3(\sec^2\theta - 1)\,d\theta$$

$$= 3\int(\sec^2\theta - 1)\,d\theta$$

$\theta = \sec^{-1}\left(\frac{x}{3}\right)$

$$= 3(\tan\theta - \theta) + C$$

$$= \boxed{3\left(\frac{\sqrt{x^2-9}}{3} - \sec^{-1}\left(\frac{x}{3}\right)\right) + C}$$

4. (12 pts). Evaluate the following integral or show that it is a divergent improper integral.

$$\int_1^4 \frac{1}{(x-2)^3}\,dx = \underbrace{\int_1^2 \frac{1}{(x-2)^3}\,dx}_{I_1} + \underbrace{\int_2^4 \frac{1}{(x-2)^3}\,dx}_{I_2}$$

$$I_1 = \lim_{t\to 2^-} \int_1^t \frac{1}{(x-2)^3}\,dx = \lim_{t\to 2^-} \int_{x=1}^{x=t} \frac{1}{u^3}\,du = \lim_{t\to 2^-} \int_{x=1}^{x=t} u^{-3}\,du$$

$u = x-2$
$du = dx$

$$= \lim_{t\to 2^-} \left.\frac{u^{-2}}{-2}\right|_{x=1}^{x=t} = \lim_{t\to 2^-} \left.-\frac{1}{2u^2}\right|_{x=1}^{x=t}$$

$$= \lim_{t\to 2^-} \left.\frac{-1}{2(x-2)^2}\right|_1^t = \lim_{t\to 2^-} \underbrace{\frac{-1}{2(t-2)^2}}_{-\infty} + \frac{1}{2(-1)^2}$$

$\therefore$ I_1 diverges

(No need to check I_2)

$$\therefore \boxed{\int_1^4 \frac{1}{(x-2)^3}\,dx \text{ Diverges}}$$

5. (8 pts). Write out the ***form only*** of the partial fraction decomposition for the following function. Do **NOT** determine the values of the coefficients.

$$\frac{3x^2-1}{x(x+2)^3(2x^2+3)} = \boxed{\frac{A}{x} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} + \frac{Ex+F}{2x^2+3}}$$

6. (8 pts). List out the first 4 terms in the following sequence.

$a_1 = 2, \;\; a_{n+1} = 2a_n - 1$

$a_1 = 2$

$a_2 = 2a_1 - 1 = 2(2) - 1 = 3$

$a_3 = 2a_2 - 1 = 2(3) - 1 = 5$

$a_4 = 2a_3 - 1 = 2(5) - 1 = 9$

7. (12 pts). Determine whether the following **sequences** converge or diverge. **If it converges, find the limit. If it diverges, clearly explain the reason why.** [Clearly indicate $+\infty$ or $-\infty$ in the case of an infinite limit.]

(a). $a_n = 3 + \underbrace{\left(-\frac{2}{3}\right)^n}_{\to 0}$

$r^n \to 0$ if $|r| < 1$

$a_n \to 3 + 0 = \boxed{3}$

(b). $b_n = \sqrt{\dfrac{1 + 4n^2}{1 + n^2}}$

$\lim_{n \to \infty} \sqrt{\dfrac{1+4n^2}{1+n^2}}$

$= \sqrt{\dfrac{4}{1}}$

$= \boxed{2}$